Honors Calculus, Math 1451, Exam I, sample Solution.

(1) (a) Given two points $P(1,3,5)$ and $Q(-1,1,1)$ and a plane $2x+3y-z=1$. To Find the intersection point of given plane and the line through P in the direction of Q, the parametric equation of the line is the one with direction vector $\overrightarrow{PQ}=(-1,1,1)-(1,3,5)$ $=\langle -2,-2,-4\rangle$ and point P: $\langle -2t+1, -2t+3, -4t+5\rangle$.

Then putting the parameters into the plane, we have

$$2(-2t+1)+3(-2t+3)-(-4t+5)=1 \Rightarrow -6t=-5, \Rightarrow t=\frac{5}{6}$$

$\Rightarrow$ the intersection point is $\left(-\frac{5}{3}+1, -\frac{5}{3}+3, -\frac{10}{3}+5\right)=\underline{\left(-\frac{2}{3},\frac{4}{3},\frac{5}{3}\right)}$

(b) Given three points $P(2,1,0)$, $Q(3,-1,1)$, $R(4,1,-1)$.

To find a plane passing P, Q, R, We have TWO vectors on this plane

$$\overrightarrow{PQ}=(3,-1,1)-(2,1,0)=(1,-2,1)$$
$$\overrightarrow{PR}=(4,1,-1)-(2,1,0)=(2,0,-1)$$

Then the normal vector $\vec{n}$ of the plane is $\overrightarrow{PQ}\times\overrightarrow{PR}=$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 2\vec{i}+3\vec{j}+4\vec{k}$$

Thus the equation of the plane is $\underline{2x+3y+4z=7}$

(1) (c) Given two vectors $<1,-3,1>$ and $<-3,1,9>$.

We have

$$\cos(\theta) = \frac{<1,-3,1>\cdot<-3,1,9>}{|<1,-3,1>|\,|<-3,1,9>|} = \frac{-3-3+9}{\sqrt{1+9+1}\,\sqrt{9+1+81}} = \frac{3}{\sqrt{11}\cdot\sqrt{91}}$$

Then $\theta = \arccos\left(\frac{3}{\sqrt{11}\cdot\sqrt{91}}\right)$

(d) Given two vectors $\vec{u} = <3,1,0>$, $\vec{v} = <1,-1,3>$.

The parallelogram spanned by $\vec{u}$ and $\vec{v}$ are

$$|\vec{u}\times\vec{v}| = \left|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ 1 & -1 & 3 \end{vmatrix}\right| = |3\vec{i} - 9\vec{j} - 4\vec{k}| = \sqrt{3^2+9^2+4^2} = \sqrt{106}$$

(2) (a) Let $\vec{r}(t)$ be a differentiable curve in $\mathbb{R}^3$. Then

$$\frac{d}{dt}(\vec{r}(t)\times\dot{\vec{r}}(t)) = \dot{\vec{r}}(t)\times\dot{\vec{r}}(t) + \vec{r}(t)\times\ddot{\vec{r}}(t)$$
$$= 0 + \vec{r}(t)\times\ddot{\vec{r}}(t).$$

(b) Let $\vec{r}(t)$ be a differentiable curve in $\mathbb{R}^3$, and $\dot{\vec{r}}(t)\cdot\vec{r}(t) = 0$. $\forall t$.

Then we have

$$\frac{d}{dt}|\vec{r}(t)|^2 = \frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t)) = \dot{\vec{r}}(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\dot{\vec{r}}(t)$$
$$= 0 + 0 = 0$$

$\Rightarrow |\vec{r}(t)|^2$ is constant.

(3) (a) Let $\overset{f(x,y)}{\overset{\shortparallel}{z}} = \frac{1}{\sqrt{x^2+y^2}}$. To find the linearization of z at $(1,1)$,

We have

$$f(x,y) \approx f(1,1) + f_x(1,1)\cdot(x-1) + f_y(1,1)\cdot(y-1)$$

$$= \frac{1}{\sqrt{2}} + \left.\frac{2x}{-2(x^2+y^2)^{\frac{3}{2}}}\right|_{(1,1)}(x-1) + \left.\frac{2y}{-2(x^2+y^2)^{\frac{3}{2}}}\right|_{(1,1)}(y-1)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}(x-1) - \frac{1}{2\sqrt{2}}(y-1) = \underline{\sqrt{2} - \frac{x+y}{2\sqrt{2}}}$$

and the estimation of $f(x,y)$ at $(1.01, 0.98)$ is

$$\sqrt{2} - \frac{1.01+0.98}{2\sqrt{2}} = \underline{\sqrt{2} - \frac{1.99}{2\sqrt{2}}}$$

(b) Let $z = \frac{1}{\sqrt{x^2+y^2}}$. To show $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, we have

$$\frac{\partial z}{\partial x} = -x(x^2+y^2)^{-\frac{3}{2}}, \quad \frac{\partial z}{\partial y} = -y(x^2+y^2)^{-\frac{3}{2}}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = -(x^2+y^2)^{-\frac{3}{2}} - x\cdot\left(-\frac{3}{2}\right)\cdot 2x(x^2+y^2)^{-\frac{5}{2}} \quad \text{and}$$

$$+ \quad \frac{\partial^2 z}{\partial y^2} = -(x^2+y^2)^{-\frac{3}{2}} - y\left(-\frac{3}{2}\right)2y(x^2+y^2)^{-\frac{5}{2}}$$

Then $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -2(x^2+y^2)^{-\frac{3}{2}} + 3(x^2+y^2)(x^2+y^2)^{-\frac{5}{2}} = 0$?

(4) (a) Let $z=f(u)+g(v)$ where $u=x+ct$, $v=x-ct$ with constant c, variables x, t.

To show $\frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}$, we have

$$\frac{\partial z}{\partial t}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial t}=\frac{\partial f}{\partial u}\cdot c+\frac{\partial g}{\partial v}(-c)$$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}\cdot 1+\frac{\partial g}{\partial v}\cdot 1$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2}=\frac{\partial}{\partial t}\left(c\frac{\partial f}{\partial u}-c\frac{\partial g}{\partial v}\right)=c\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial t}-c\frac{\partial^2 g}{\partial v^2}\cdot\frac{\partial v}{\partial t}$$

$$=c^2\frac{\partial^2 f}{\partial u^2}+c^2\frac{\partial^2 g}{\partial v^2} \quad \text{and}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 g}{\partial v^2}\cdot\frac{\partial v}{\partial x}=\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}$$

Then we have $\frac{\partial^2 z}{\partial t^2}=c^2\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}\right)=c^2\frac{\partial^2 z}{\partial x^2}$

(b) It is correct, by def. of partial derivative of f w.r.t x,

we have

$$\frac{\partial f}{\partial x}(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}.$$

So as we want to find $f_x(a,b,c)$

we have $$\frac{\partial f}{\partial x}(a,b,c)=\lim_{\Delta x\to 0}\frac{f(a+\Delta x,b,c)-f(a,b,c)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{g(a+\Delta x)-g(a)}{\Delta x}=\frac{dg}{dx}(a)$$