Honors Calculus, Sample Exam Questions

(1) Calculate $\int_C f ds$ where $f(x, y) = xy$ and C is the part of the parabola $y = x^2$ between $(0, 0)$ and $(1, 1)$.

(2) Show that

$$\mathrm{div}(\mathrm{grad}(f) \times \mathrm{grad}(g)) = 0$$

if $f,\ g : \mathbb{R}^3 \to \mathbb{R}$ are functions with continuous second order partial derivatives.

(3) (a) Find a parametric representation for the surface $x^2 + y^2 = 9$ that lies between the planes $z = 0$ and $z = 4$.

(b) Find the equation of the tangent plane to the given surface at the point $(3, 0, 1)$.

(4) Calculate $\int_C f ds$ where $f(x, y) = y \cos x$ and C is the part of the parabola $y = \sin x$ between $(0, 0)$ and $(\frac{\pi}{2}, 1)$.

(5) Calculate $\int_C P dx + Q dy$ where $P(x, y) = y^2$, $Q(x, y) = x^2$ and C is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(1, 0)$ oriented positively.

(6) Let $F(x, y, z) = (y^2, 2xy + z, y)$.

(a) Show that the integral of F between any two points is independent of the curve between the two points. State precisely any theorems or results that you use.

(b) Using (a) or otherwise find $\int_C F \cdot dr$ where C is a straight line segment connecting $P = (0, 0, 0)$ to $Q = (2, 1, 1)$.

(7) Suppose that f, g are differentiable vector-valued functions $f,\ g : \mathbb{R} \to \mathbb{R}^3$. Show

$$\frac{d}{dt}(f \cdot g) = (\frac{d}{dt} f) \cdot g + f \cdot (\frac{d}{dt} g)$$

Hence show that if $|f(t)| = 1$ for all t then $f(t)$ is orthogonal to $\frac{d}{dt} f(t)$ for all t.

(8) Find the work done by the force field $F(x, y, z) = (z, x, y)$ in moving a particle from the point $(1, 0, 0)$ to $(0, \pi/2, 1)$ along:

(a) a straight line.

(b) the helix $r(t) = (\cos t, t, \sin t)$.

(9) Consider the vector field $F(x, y, z) = xy^2\hat{\imath} + x\hat{\jmath} + \frac{z^3}{3}\hat{k}$ and the surface S described by the cylinder $\{(x, y, z) : y^2 + z^2 = 1,\ 0 \leq x \leq 2\}$.

(a) Sketch the surface S.

(b) Calculate the surface integral $\int\int_S F \cdot n \, dS$ where n is the unit normal vector to the surface S by using the Divergence Theorem or otherwise. Make sure to state an appropriate version of the Divergence Theorem in your answer.

(10) Show that if A is a region which is the interior of a simple closed cure C oriented counterclockwise then

$$\text{Area}(A) = \frac{1}{2} \int_C F(x, y) \cdot dr$$

if $F(x, y) = (-y, x)$.

(11) Suppose $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field given by $F(x, y, z) = (2x, z^2, 2yz)$ and $P = (0, 1, 0)$, $Q = (1, 3, -1)$.

(a) Show that the integral of the vector field F along a curve C joining P to Q is independent of the curve C.

(b) Using (a) or otherwise find $\int_C F \cdot dr$ where C is a straight-line segment joining P to Q.

(12) (a) Find the length of the spiral-shaped curve $r(t) = (2\cos t, 2\sin t, 3t)$ between $t = 0$ and $t = \pi$.

(b) Find the surface area of the paraboloid in $\mathbb{R}^3$ given by

$$z = x^2 + y^2 \qquad 0 \leq z \leq 1$$

(13) (a) Parametrize that part, call it S, of the unit sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant by spherical coordinates (ρ, ϕ, θ). (b) Find the flux of the vector field $F(x, y, z) = (0, 0, z^2)$ across the surface S in (a).

(14) Consider the vector field $F(x, y, z) = xy^2\hat{\imath} + x\hat{\jmath} + \frac{z^3}{3}$ and the surface S described by the cylinder $\{(x, y, z) : y^2 + z^2 = 1,\ 0 \leq x \leq 2\}$.

(a) Sketch the surface S.

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(15) Use Stokes theorem to evaluate $\int_S curlF \cdot ndS$ where $F(x, y, z) = 2y\cos(z)\hat{\imath} + e^x \sin(z)\hat{\jmath} + xe^y\hat{k}$ where S is the hemisphere $x^2+y^2+z^2 = 9$, $z \geq 0$, oriented positively.