Honors Calculus, Sample Final Exam Questions

(1) Since $\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$, so $\int \vec{F}\cdot\vec{T}\,ds = \int \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}'(t)}{|\vec{r}'(t)|}|\vec{r}'(t)|\,dt$

$$= \int \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

(a) Given $F(x,y) = (x^2, xy)$ and curve C be the part of the parabola between $(0,0)$ and $(1,1)$ (Assume C is $y=x^2$)

Let $x=t$, $y=t^2$ where $0\le t\le 1$. Then $\vec{r}(t) = <t, t^2>$, $\vec{r}'(t) = <1, 2t>$

and $\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_0^1 <t^2, t^3>\cdot<1, 2t>\,dt$

$$= \int_0^1 t^2 + 2t^4\,dt = \frac{t^3}{3} + \frac{2}{5}t^5\Big|_0^1 = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}$$

(b) Given $F(x,y) = (x^2, y^2)$ and curve C be the part of $y = \sin(x)$ where $0\le x\le \pi$.

Let $x=t$, $y=\sin(t)$ where $0\le t\le\pi$. Then $\vec{r}(t) = <t, \sin(t)>$

$\vec{r}'(t) = <1, \cos(t)>$ and $dx = dt$

$$\int_C \vec{F}\cdot\vec{T}\,dx = \int_0^\pi \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}'(t)}{|\vec{r}'(t)|}\,dt = \int_0^\pi \frac{<t^2, \sin^2(t)>\cdot<1, \cos(t)>}{\sqrt{1+\cos^2(t)}}\,dt$$

$$= \int_0^\pi \left(t^2 + \cos(t)\cdot\sin^2(t)\right)\frac{dt}{\sqrt{1+\cos^2(t)}}$$

(c) Given $P(x,y) = y^2$, $Q(x,y) = -x$ and curve C be $x = \frac{y^2}{4}$ from $(0,0)$ to $(1,2)$. Let $y=t$ and $x=\frac{t^2}{4}$ where $0\le t\le 2$, Then $\vec{r}(t) = <\frac{t^2}{4}, t>$, $\vec{r}'(t) = <\frac{t}{2}, 1>$ and

$$\int_C P\,dx + Q\,dy = \int_0^2 <P(\vec{r}(t)), Q(\vec{r}(t))>\cdot\vec{r}'(t)\,dt$$

$$= \int_0^2 <t^2, -\frac{t^2}{4}>\cdot<\frac{t}{2}, 1>\,dt = \int_0^2 \frac{t^3}{2} - \frac{t^2}{4}\,dt = \frac{t^4}{8} - \frac{t^3}{12}\Big|_0^2 = 2 - \frac{2}{3} = \frac{4}{3}$$

(2) Given $F(x,y,z) = \langle yz + y\cos(xy),\ xz + x\cos(xy),\ xy \rangle$

(a) Showing $\vec{F}$ is path independent is sufficiently showing there is a scalar function f such that $\nabla f = \vec{F}$.

Now, we check curl $\vec{F}$, since curl with a gradient of a scalar function is zero (curl $(\nabla f) = 0$), if curl $\vec{F} = 0$. Then we are done. Thus $\left(\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle\right)$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+y\cos(xy) & xz+x\cos(xy) & xy \end{vmatrix}$$
(curl $\vec{F}$)

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz + \cos(xy))\right)\vec{i} - \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz + y\cos(xy))\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}(xz + x\cos(xy)) - \frac{\partial}{\partial y}(yz + y\cos(xy))\right)\vec{k}$$

$$= (x - x)\vec{i} - (y - y)\vec{j} + \left[(z + \cos(xy) - xy\sin(xy)) - (z + \cos(xy) - xy\sin(xy))\right]\vec{k}$$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$ which means $\vec{F}$ is path independent.

(b) Since $\vec{F}$ is path independent, So we can do line integral between $P(0,0,0)$ to $Q(\pi,1,0)$ by a segment between them.

Let $\vec{r}(t)$ be this segment, we have $\vec{r}(t) = (0,0,0) + ((\pi,1,0) - (0,0,0))t$
$= \langle \pi t, t, 0 \rangle$ where $0 \le t \le 1$, and $\vec{r}'(t) = \langle \pi, 1, 0 \rangle$.

Then $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle t\cos(\pi t^2), \pi t\cos(\pi t^2), \pi t^2 \rangle \cdot \langle \pi, 1, 0 \rangle \, dt$

$$= \int_0^1 2\pi t\cos(\pi t^2)\, dt = \frac{\sin(\pi t^2)}{2}\Big|_0^1 = 0$$

(3) Given two scalar functions $f, g: \mathbb{R}^3 \to \mathbb{R}$ with continuous second order partial derivatives. Then we have

$$\operatorname{grad}(f) = \langle f_x, f_y, f_z \rangle, \quad \operatorname{grad}(g) = \langle g_x, g_y, g_z \rangle.$$

and

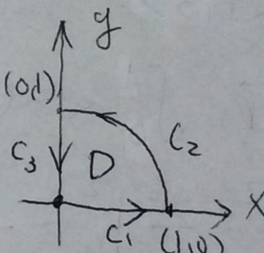
$$\operatorname{div}(\operatorname{grad}(f) \times \operatorname{grad}(g)) = \operatorname{div} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix}$$

$$(\nabla \cdot = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot)$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \frac{\partial}{\partial x}(f_y g_z - f_z g_y) - \frac{\partial}{\partial y}(f_x g_z - f_z g_x) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x)$$

$$= f_{xy} g_z + f_y g_{xz} - f_{xz} g_y - f_z g_{xy} - f_{xy} g_z - f_x g_{yz} + f_{yz} g_x + f_z g_{xy} + f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz} = 0$$

(4) Given $\vec{F} = (x, x^2 + xy)$ and the curve C:



So the work $W = \oint_C \vec{F} \cdot d\vec{r}$. By Green's Theorem, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial(x^2+xy)}{\partial x} - \frac{\partial(x)}{\partial y} \, dA = \iint_D 2x + y \, dA$$

$$= \iint_D 2x + y \, dA \underset{\uparrow}{=} \int_0^1 \int_0^{\frac{\pi}{2}} (2r\cos\theta + r\sin\theta) r \, d\theta \, dr = \int_0^1 \int_0^{\frac{\pi}{2}} 2r^2\cos(\theta) + r^2 \sin(\theta) \, d\theta \, dr$$

$x = r\cos(\theta)$
$y = r\sin(\theta)$
$0 \le \theta \le \frac{\pi}{2}$
$0 \le r \le 1$

$$= \frac{2r^3}{3}\Big|_0^1 \cdot \sin\theta \Big|_0^{\frac{\pi}{2}} + \frac{r^3}{3}\Big|_0^1 \left[-\cos(\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 1 = 1$$

(5)

(a) Given $f(x,y)=4xy$ and curve C be a line segment between $(-1,-1)$ and $(2,1)$. The equation of this line is $y-1=\frac{2}{3}(x-2)$

$$\Rightarrow y=\frac{2}{3}x-\frac{1}{3}$$

Then $\int_C f\,ds=\int_{-1}^{2} 4x\left(\frac{2}{3}x-\frac{1}{3}\right)\cdot\sqrt{1+\left(\frac{2}{3}\right)^2}\,dx$

$$=\frac{\sqrt{13}}{3}\cdot 4\int_{-1}^{2}\frac{2}{3}x^2-\frac{x}{3}\,dx=\frac{4\sqrt{13}}{3}\left[\frac{2x^3}{9}-\frac{x^2}{6}\right]_{-1}^{2}$$

$$=\frac{4}{3}\sqrt{13}\left(2-\frac{1}{2}\right)=2\sqrt{13}.$$

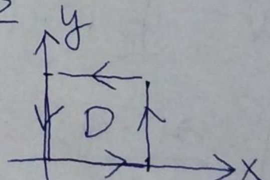
(b) Given $\vec{F}(x,y)=\langle -x,y^2\rangle$ and curve C be part of $y=x^2$ ~~between~~ from $(1,1)$ to $(2,4)$. Let $x=t$, $y=t^2$ and $\vec{r}(t)=\langle t,t^2\rangle$ where $1\le t\le 2$

$\vec{r}'(t)=\langle 1,2t\rangle$ Then.

$$\int_C \vec{F}\cdot d\vec{r}=\int_1^2\langle -t,t^4\rangle\cdot\langle 1,2t\rangle\,dt=\int_1^2 -t+2t^5\,dt$$

$$=-\frac{t^2}{2}+\frac{t^6}{3}\Big|_1^2=-\frac{3}{2}+21=\frac{39}{2}$$

(c) Given $\vec{F}=\langle y^2,x\rangle$ and curve C :



We have $\int_C \vec{F}\cdot d\vec{r}=\iint_D \frac{\partial(x)}{\partial x}-\frac{\partial}{\partial y}(y^2)\,dA=\iint_D 1-2y\,dA$

$$=\int_0^1\int_0^1 1-2y\,dy\,dx=\int_0^1 y-y^2\Big|_0^1 dx=0$$

(6)

(a) If $\vec{F}(x,y,z) = \nabla f(x,y,z)$, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)dt = \int_a^b \nabla f(\vec{r}(t))\cdot \vec{r}'(t)dt.$$

Since $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t))\cdot \vec{r}'(t)$, then we obtain

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t))\cdot \vec{r}'(t)dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))dt$$

$$\underset{\uparrow}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

By Fundamental thm. of Calculus

(b) Given $\vec{F}(x,y,z) = (yz+e^{-y}-ye^{-x},\ xz+e^{-y}-xe^{-y},\ xy)$

To check the existence of f, we have

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+e^{-y}-ye^{-x} & xz+e^{-y}-xe^{-y} & xy \end{vmatrix} = (x-x)\vec{i} - (y-y)\vec{j} + (z-e^{-y}-z+e^{-y}-e^{-x})\vec{k}$$

$$= -e^{-x}\vec{k} \neq 0.$$

So we cannot find f such that $\vec{F} = \nabla f$.

(7) (a) Given $\vec{r} = (x, y, z)$. and $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

(i)
$$\nabla \frac{1}{|\vec{r}|} = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \left(-\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{1}{2}\frac{2y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{1}{2}\frac{2z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$$

$$\downarrow$$

$$(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= \left(-\frac{x}{(\sqrt{x^2+y^2+z^2})^3}, -\frac{y}{(\sqrt{x^2+y^2+z^2})^3}, -\frac{z}{(\sqrt{x^2+y^2+z^2})^3}\right) = -\frac{\vec{r}}{|\vec{r}|^3}$$

(ii)
$$\text{curl}(\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\vec{i} - \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)\vec{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\vec{k} = 0$$

(b) By (a), we have

Since $\vec{F} = -\frac{r}{|\vec{r}|^3} = \nabla \frac{1}{|\vec{r}|}$, Then

$$\int_C \vec{F} \cdot d\vec{r} = \left.\frac{1}{|\vec{r}(t)|}\right|_0^{2\pi} = \frac{1}{\sqrt{1+4\pi^2}} - \frac{1}{\sqrt{1+0^2}} = \frac{1}{\sqrt{1+4\pi^2}} - 1$$

$$\uparrow$$

$c(t) = (\cos t, \sin t, t)$, $0 < t \le 2\pi$

$\Rightarrow |\vec{r}| = \sqrt{\cos^2(t) + \sin^2(t) + t^2} = \sqrt{1+t^2}$. $0 \le t \le 2\pi$

(8)

(a) Green's theorem

Let A be a domain which is the interior of a closed curve C, oriented anti-clockwise. If P and Q have continuous partial derivatives on an open region that contains A, then $\oint_C Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

Using Green's theorem, the area of A is

$$\iint_A 1\, dxdy \quad \text{and} \quad \oint_C x\, dy = \iint_A \frac{\partial x}{\partial x} - 0\, dxdy = \iint_A 1\, dxdy.$$

$$\Rightarrow \text{area of } A = \oint_C x dy$$

(b) If A is an ellipse with boundary C: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. By (a), we have

$$\boxed{\begin{array}{l} x = a\cos(\theta) \\ y = b\sin(\theta) \end{array} \quad 0 \le \theta \le 2\pi. \quad dy = b\cos(\theta)d\theta}$$

$$\text{area of } A = \oint_C x dy = \int_0^{2\pi} a\cos(\theta) \cdot b\cos(\theta) d\theta$$

$$= ab \int_0^{2\pi} \cos^2(\theta) d\theta = ab \cdot \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} d\theta = ab\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{2\pi}$$

$$= \pi ab.$$

(9) (a) $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

Given $\vec{F}(x,y,z) = \langle x^2, xyz, z^2 \rangle$, we have

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle x^2, xyz, z^2 \rangle$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(z^2) = 2x + xz + 2z.$$

and

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xyz & z^2 \end{vmatrix} = \left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(xyz)\right)\vec{i} - \left(\frac{\partial (z^2)}{\partial x} - \frac{\partial (x^2)}{\partial z}\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(x^2)\right)\vec{k}$$

$$= (0 - xy)\vec{i} - (0 - 0)\vec{j} + (yz - 0)\vec{k} = (-xy)\vec{i} + 0\vec{j} + (yz)\vec{k}.$$

(b) If $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, let $\vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

We have

$$\text{div}\,(\text{curl}\,\vec{F}) = \text{div} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \rangle$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial x \partial y} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial y \partial z} = 0$$

(9)

(c) Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2+y^2+z^2$, we have

$\nabla f = \langle 2x, 2y, 2z \rangle$ and

$\mathrm{div}(\nabla f) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z) = 2+2+2 = 6 \neq 0$.

(10)

See Q(4)