Honors Calculus, Sample Final Exam Questions.

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(1) Calculate the following line integrals:

(a) $\int_C F \cdot T ds$ where $F(x,y) = (x^2, xy)$, C is the part of the parabola between $(0,0)$ and $(1,1)$ and T is the unit tangent vector to C.

(b) $\int_C F \cdot T dx$ where $F(x,y) = (x^2, y^2)$, C is the part of the curve $y = \sin x$ for $0 \leq x \leq \pi$ and T is the unit tangent vector to C.

(c) $\int_C P dx + Q dy$ where $P(x,y) = y^2$, $Q(x,y) = -x$ and C is the part of the parabola $x = y^2/4$ from $(0,0)$ to $(1,2)$.

(2) Let $F(x,y,z) = (yz + y\cos(xy), xz + x\cos(xy), xy)$.

(a) Show that the integral of F between any two points is independent of the curve between the two points.

(b) Find $\int_C F \cdot dr$ where C is a curve connecting $P = (0,0,0)$ to $Q = (\pi, 1, 0)$.

(3) Show that

$$\mathrm{div}(\mathrm{grad}(f) \times \mathrm{grad}(g)) = 0$$

if $f, g : \mathbb{R}^3 \to \mathbb{R}$ are functions with continuous second order partial derivatives.

(4) A particle starts at the point $(0,0)$ and moves along the x axis to $(1,0)$, then along that part of the circle $x^2 + y^2 = 1$ between $(1,0)$ and $(0,1)$ and then along the y to $(0,0)$ again. Use Green's Theorem to find the work done on the particle by the force field

$$F(x,y) = (x, x^2 + xy)$$

by traversing this curve.

(5) Calculate the following line integrals:

(a) $\int_C f ds$ where $f(x,y) = 4xy$ and C is the straight line segment between $(-1,-1)$ and $(2,1)$.

(b) $\int_C F \cdot dr$ where $F(x,y) = (-x, y^2)$ and C is the part of the curve $y = x^2$ between $(1,1)$ and $(2,4)$.

(c) $\int_C F \cdot dr$ where $F = y^2\hat{i} + x\hat{j}$ and C is the boundary of the unit square in the plane with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ parametrized anti-clockwise.

(6) (a) If $r(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$ is a parametrized curve in $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function show, using the chain rule, that $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot \dot{r}(t)$. Hence show that if $F = \nabla f$ then $\int_C F \cdot dr = f(r(b)) - f(r(a))$ where C is parametrized by $r(t)$, $a \leq t \leq b$.

(b) Let $F(x,y,z) = (yz + e^{-y} - ye^{-x}, xz + e^{-y} - xe^{-y}, xy)$. Find a function $f(x,y,z)$ such that $\nabla f = F$. Explain why the result of (a) shows that the integral of F between any two points is independent of the curve joining the two points.

(7) (a) Let $r = (x, y, z)$ denote the position vector of a point in $\mathbb{R}^3$ with length $|r| = \sqrt{x^2 + y^2 + z^2}$. Show that:

(i) $\nabla \frac{1}{|r|} = -\frac{r}{|r|^3}$

(ii) $\text{curl}(r) = 0$.

(b) Let $F(r) = -\frac{r}{|r|^3}$ be a vector field in $\mathbb{R}^3$. Find $\int_C F \cdot dr$ where C is the helix $C(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$.

(8) Let A be a domain which is the interior of a closed curve C oriented anti-clockwise.

(a) State Green's theorem. Use Green's theorem to show that the area of A is given by $\int_C x \, dy$.

(b) Use the result from (a) to show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ has area πab. *Hint: a suitable parametrization for the ellipse is* $x = a\cos\theta, y = b\sin\theta$, $0 \leq \theta \leq 2\pi$.

(9) (a) Find the divergence and curl of the vector field:

$$F(x, y, z) = (x^2, xyz, z^2)$$

(b) If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field does $\text{div}(\text{curl } F) = 0$? Either prove or give a counterexample.

(c) If $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function does $\text{div}(\nabla f) = 0$? Either prove or give a counterexample.

(10) A particle starts at the point $(0,0)$ and moves along the x axis to $(1,0)$, then along that part of the circle $x^2+y^2=1$ between $(1,0)$ and $(0,1)$ and then along the y axis to $(0,0)$ again. Use Green's Theorem to find the work done on the particle by the force field

$$F(x,y)=(x,x^2+xy)$$

by traversing this curve. *Hint: use polar coordinates.*