Honors Calculus, Exam 2 Practice 1

(1) Given $f(x,y,z)=x^2+y^2-4z$ at $(1,1,-1)$ and $\vec{u}=\frac{1}{\sqrt{5}}(2,1,0)$.

The direction of max. increase of f at $(1,1,-1)$ is

$$\nabla f(1,1,-1)=\langle 2x,2y,-4\rangle\Big|_{(1,1,-1)}=\langle 2,2,-4\rangle.$$

and the directional derivative of f in $\vec{u}$ is

$$D_{\vec{u}}f\Big|_{(1,1,-1)}=\nabla f\cdot\vec{u}\Big|_{(1,1,-1)}=\langle 2,2,-4\rangle\cdot\langle\tfrac{2}{\sqrt{5}},\tfrac{1}{\sqrt{5}},0\rangle=\frac{6}{\sqrt{5}}$$

(2)(a) Given $f(x,y)=2x+3y-x^2-y^2$ on closed square with vertices $(0,0)$, $(0,2)$, $(2,0)$ and $(2,2)$.

To Find the max and min value of f

(I) First, find the critical point, we have.

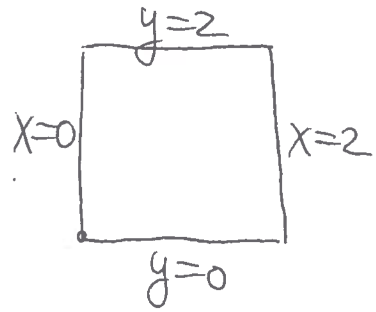
$$f_x(x,y)=2-2x=0,\ f_y(x,y)=3-2y=0 \Rightarrow \text{critical pt is } (1,\tfrac{3}{2})$$

Using Second derivatives Test, we have.

$$\underset{<0}{f_{xx}=-2}\quad f_{xy}=0,\quad f_{yy}=-2 \Rightarrow D=f_{xx}f_{yy}-[f_{xy}]^2=4>0$$

$$\Rightarrow f \text{ has a local max } f(1,\tfrac{3}{2})=2+\tfrac{9}{2}-1-\tfrac{9}{4}=\tfrac{13}{4}.$$

(II) Second, checking the boundaries



$\triangle$ on $x=0$, $f(0,y)=3y-y^2$, $0\le y\le 2$,

$$\frac{df}{dy}(0,y)=3-2y=0,\quad y=\frac{3}{2}.$$

$f(0,\frac{3}{2})=\frac{9}{2}-\frac{9}{4}=\frac{9}{4}$ is the local max.

$f(0,0)=0$ is the local min.

△ $x=2$, $f(2,y)=3y-y^2$, $0\le y\le 2$ (same function)

The same case of the previous one:

$f(2,\frac{3}{2})$ is the local max and $f(2,0)=0$ is the local min.

△ $y=0$. $f(x,0)=2x-x^2$, $0\le x\le 2$.

$\frac{df}{dx}(x,0)=2-2x=0$, $x=1$.

$f(1,0)=1$ is the local max.

$f(0,0)=f(2,0)=0$ is the local min.

△ $y=2$, $f(x,2)=2x-x^2+2$, $0\le x\le 2$.

$\frac{df}{dx}(x,2)=2-2x$, $x=1$.

$f(1,2)=3$ is the local max

$f(0,2)=f(2,2)=2$ is the local min.

By part (I) & (II).

f has abs. max $\frac{13}{4}$ at $(1,\frac{3}{2})$ and abs. min 0 at $(0,0)$, $(2,0)$.

(2)(b) Given $f(x,y)=4+3xy-y^2=0$ and $g(x,y)=x^2+y^2$.

Find the max of g subject to $f=0$. by Lagrange Multiplier,

We have $\begin{cases}\nabla g=\lambda\nabla f\\ f=0\end{cases} \Rightarrow \begin{cases}\langle 2x,2y\rangle=\lambda\langle 3y,3x-2y\rangle\\ 4+3xy-y^2=0\end{cases}$

$\begin{cases}2x=3\lambda y & \text{—(1)}\\ 2y=\lambda(3x-2y) & \text{—(2)}\\ 4+3xy-y^2=0 & \text{—(3)}\end{cases} \Rightarrow \begin{matrix}\lambda=\frac{2x}{3y}\\ \lambda=\frac{2y}{3x-2y}\end{matrix} \Rightarrow \frac{2x}{3y}=\frac{2y}{3x-2y}$

$\Rightarrow 2x(3x-2y)=6y^2$

$\Rightarrow 6x^2-4xy-6y^2=0$ —(4)

$\Rightarrow x=\frac{1\pm\sqrt{10}}{3}y$

$\Rightarrow \begin{cases}6x^2-4xy-6y^2=0\\ 4+3xy-y^2=0\end{cases} \Rightarrow$

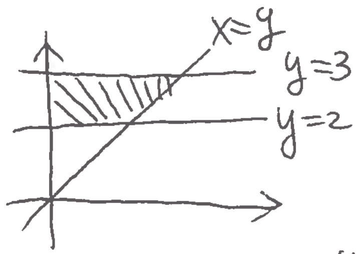
$\Rightarrow$ As $x=\frac{1+\sqrt{10}}{3}y$, $4+3\left(\frac{1+\sqrt{10}}{3}\right)y^2-y^2=0 \Rightarrow 4=-\sqrt{10}y^2$, a contradiction,

$\Rightarrow$ As $x=\frac{1-\sqrt{10}}{3}y$, $4+3\left(\frac{1-\sqrt{10}}{3}\right)y^2-y^2=0 \Rightarrow 4=\sqrt{10}y^2$.

$\Rightarrow y^2=\frac{4}{\sqrt{10}} \Rightarrow y=\pm\frac{2}{\sqrt[4]{10}}$ so $x=\frac{1-\sqrt{10}}{3}\cdot\pm\frac{2}{\sqrt[4]{10}}=\pm\frac{2}{3}\left(\frac{1-\sqrt{10}}{\sqrt[4]{10}}\right)$.

$$g\left(\frac{2}{3}\left(\frac{1-\sqrt{10}}{\sqrt[4]{10}}\right),\frac{2}{\sqrt[4]{10}}\right)=\left(\frac{11-2\sqrt{10}}{9}+1\right)\frac{4}{\sqrt{10}}=\left(\frac{20-2\sqrt{10}}{9}\right)\frac{4}{\sqrt{10}}$$

(3) $\iint_R x^2y^2\,dA$, $R=$



$=\{(x,y)\mid 2\le y\le 3,\ 0\le x\le y\}$

$$\Rightarrow \int_2^3\int_0^y x^2y^2\,dx\,dy=\int_2^3 \frac{x^3}{3}y^2\Big|_0^y\,dy=\int_2^3\frac{y^5}{3}\,dy$$

$$=\frac{y^6}{18}\Big|_2^3=\frac{1}{18}(3^6-2^6)=\frac{665}{18}.$$

(4) (a) $$\iint_D f(x,y)\,dx\,dy \underset{\substack{x=r\cos\theta\\ y=r\sin\theta}}{=\!=} \int_0^{2\pi}\int_0^{\sqrt{R}}\frac{1}{r^2+2}\,r\,dr\,d\theta$$

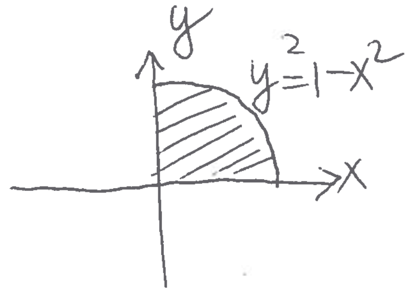
$$=\int_0^{2\pi}\frac{1}{2}\ln(r^2+2)\Big|_0^{\sqrt{R}}\,d\theta=2\pi\,\frac{1}{2}\left[\ln(R+2)-\ln(2)\right]$$

$$=\pi\left(\ln(R+2)-\ln(2)\right]$$

(b) As R tends to ∞, we get $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{x^2+y^2+2}\,dx\,dy$ Doesn't exist.

(5).

The volume is $\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{2-x^2-y^2}\,dy\,dx$

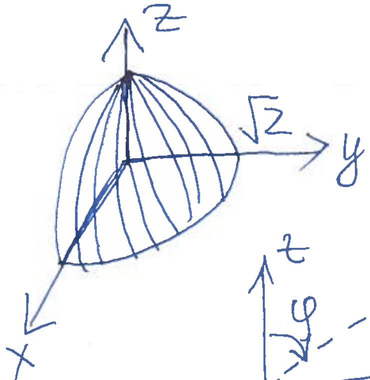


$$x=r\cos\theta,\quad y=r\sin\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{2-r^2}\, r\,dr\,d\theta$$

$$= \frac{\pi}{2}\left[-\frac{1}{3}(2-r^2)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{2}\left[-\frac{1}{3}(1-2\sqrt{2})\right]$$

$$= \frac{\pi}{6}(2\sqrt{2}-1).$$

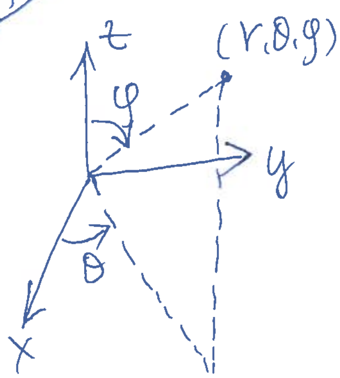


(6) Let $x=r\sin\varphi\cos\theta$

$y=r\sin\varphi\sin\theta$

$z=r\cos\varphi$

and $0\le r\le\sqrt{2}$, $0\le\theta\le\frac{\pi}{2}$, $0\le\varphi\le\frac{\pi}{2}$.



Then $\iiint_R \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$

$$= \int_0^{\sqrt{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\cancel{r^2}}\cdot\cancel{r^2}\sin\varphi\,d\varphi\,d\theta\,dr$$

$$= \int_0^{\sqrt{2}}\int_0^{\frac{\pi}{2}} -\cos\varphi\Big|_0^{\frac{\pi}{2}}\,d\theta\,dr = \sqrt{2}\cdot\frac{\pi}{2}\cdot 1 = \frac{\sqrt{2}}{2}\pi.$$