Another Final Brief Solutions

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There may be some mistakes in the following contents. Please do them by yourself and compare. If there is any question, let me know. Work hard, try to ace your final exam!

1. $y = x^2, ds = \sqrt{1+(y')^2}dx = \sqrt{1+4x^2}dx$

$$\int_C f ds = \int_0^1 x \cdot x^2 \cdot \sqrt{1+4x^2}dx$$

let $u = 1+4x^2$, then $du = 8xdx$, and $x^2 = \frac{u-1}{4}$, $u(0) = 1, u(1) = 5$

$$\int_0^1 x \cdot x^2 \cdot \sqrt{1+4x^2}dx = \int_1^5 \frac{u-1}{4}\sqrt{u} \cdot \frac{1}{8}du = \frac{1}{32}\int_1^5 u^{3/2} - u^{1/2}du$$

$$= \frac{1}{32}(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2})\Big|_1^5 = \frac{5\sqrt{5}}{24} + \frac{1}{120}$$

2.

$$grad(f) \times grad(g) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = (f_y g_z - f_z g_y)\mathbf{i} - (f_x g_z - f_z g_x)\mathbf{j} + (f_x g_y - f_y g_x)\mathbf{k}$$

$$\begin{aligned} div(grad(f) \times grad(g)) &= (f_y g_z - f_z g_y)_x - (f_x g_z - f_z g_x)_y + (f_x g_y - f_y g_x)_z \\ &= f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx} + f_{zy} g_x + f_z g_{xy} \\ &\quad - f_{xy} g_z - f_x g_{zy} + f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz} \\ &= 0 \end{aligned}$$

3. (a) $x = 3\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$, $0 \le z \le 4$

   (b) This is part of cylinder, let $f(x,y,z) = x^2+y^2-9$, $\nabla f = (2x, 2y, 0)$, so $\nabla f(3,0,1) = (6,0,0)$, the equation of the tangent plane is $6(x-3)+0(y-0)+0(z-1) = 0$, that is, the plane $x = 3$.

4. $y = \sin x$, $ds = \sqrt{1+(y')^2}\ dx = \sqrt{1+\cos^2 x}\ dx$,

$$\int_C f ds = \int_0^{\pi/2} \sin x \cos x \sqrt{1+\cos^2 x}\ dx = -\frac{1}{2}\int_1^0 \sqrt{1+u}\ du$$

$$= \frac{1}{2}\int_0^1 \sqrt{1+u}\ du = \frac{1}{3}(1+u)^{3/2}\Big|_0^1 = \frac{2\sqrt{2}-1}{3}$$

where we used u-substitution $u = \cos^2 x$, $du = -2\cos x \sin x dx$.

5.

$$\int_C Pdx + Qdy = \int_C y^2dx + x^2dy = \iint_D (2x - 2y)dxdy$$

$$= \int_0^1 \int_0^x (2x - 2y)dydx = \int_0^1 (2xy - y^2)\Big|_{y=0}^{y=x} dx = \int_0^1 x^2 dx = \frac{1}{3}$$

6. (a) If we can show $curl(\mathbf{F}) = 0$, then there exists a function f such that $\mathbf{F} = \nabla f$,

$$curl(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z & y \end{vmatrix} = (1-1)\mathbf{i} - (0-0)\mathbf{j} + (2y - 2y)\mathbf{k} = \mathbf{0}$$

so the integral of $\mathbf{F}$ between any two points is independent of the curve between the two points. (State the theorems that you think it should be)

(b) by part(a) , we know there exists a function f such that $\mathbf{F} = \nabla f$,

$$\begin{aligned} f_x &= y^2 \\ f_y &= 2xy + z \\ f_z &= y \end{aligned}$$

Integrate f_x with respect to x, we get $f(x, y, z) = xy^2 + g(y, z)$, then differentiate with respect to y and z, $f_y = 2xy + g_y = 2xy + z, f_z = g_z = y$.

$$g_y = z, g_z = y$$

so $g(y, z) = yz + C$, where C is a constant, we can take $C = 0$. Thus $f(x, y, z) = xy^2 + yz$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 1, 1) - f(0, 0, 0) = 2 + 1 - 0 = 3$$

7. let $f(t) = (x(t), y(t), z(t))$, $g(t) = (a(t), b(t), c(t))$

$$\begin{aligned} \frac{d}{dt}(f \cdot g) &= \frac{d}{dt}[x(t)a(t) + y(t)b(t) + z(t)c(t)] \\ &= x'(t)a(t) + x(t)a'(t) + y'(t)b(t) + y(t)b'(t) + z'(t)c(t) + z(t)c'(t) \\ &= [x'(t)a(t) + y'(t)b(t) + z'(t)c(t)] + [x(t)a'(t) + y(t)b'(t) + z(t)c'(t)] \\ &= (x'(t), y'(t), z'(t)) \cdot (a(t), b(t), c(t)) + (x(t), y(t), z(t)) \cdot (a'(t), b'(t), c'(t)) \\ &= (\frac{d}{dt}f) \cdot g + f \cdot (\frac{d}{dt}g) \end{aligned}$$

if $|f(t)| = 1$ for all t, then $f \cdot f = |f(t)|^2 = 1$, differentiate both sides with respect to t,

$$(\frac{d}{dt}f) \cdot f + f \cdot (\frac{d}{dt}f) = 0$$

$$2f \cdot (\frac{d}{dt}f) = 0 \implies f \cdot (\frac{d}{dt}f) = 0$$

this is true for all t, so $f(t)$ is orthogonal to $\frac{d}{dt}f(t)$ for all t.

8. (a) the parametric equation is:

$$\begin{aligned} x &= 1 + (0-1)t = 1-t \\ y &= 0 + (\pi/2 - 0)t = \frac{\pi}{2}t \\ z &= 0 + (1-0)t = t \end{aligned}$$

where $0 \le t \le 1$.

$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = (t, 1-t, \frac{\pi}{2}t) \cdot (-1, \frac{\pi}{2}, 1) = -t + \frac{\pi}{2} - \frac{\pi}{2}t + \frac{\pi}{2}t = -t + \frac{\pi}{2}$$

$$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = \int_0^1 (-t + \frac{\pi}{2})dt = (-\frac{1}{2}t^2 + \frac{\pi}{2}t)\Big|_0^1 = \frac{\pi - 1}{2}$$

(b) $\mathbf{r}(t) = (\cos t, t, \sin t)$, $d\mathbf{r} = (-\sin t, 1, \cos t)$, $(1,0,0) \to t = 0$, $(0, \pi/2, 1) \to t = \pi/2$,

$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = (\sin t, \cos t, t) \cdot (-\sin t, 1, \cos t) = \sin^2 t + \cos t + t\cos t$$

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \int_0^{\pi/2} \sin^2 t + \cos t + t\cos t \; dt \\ &= (\frac{1}{2}t - \frac{1}{4}\sin(2t) + \sin t + t\sin t + \cos t)\Big|_0^{\pi/2} \\ &= (\frac{\pi}{4} + 1 + \frac{\pi}{2}) - (0 + 1) = \frac{3}{4}\pi \end{aligned}$$

9. (a) I believe you can sketch it!

(b) let E denote the solid cylinder, enclosed by $y^2 + z^2 = 1$, $0 \le x \le 2$,

$$\int_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \nabla \cdot \mathbf{F} dV = \iiint_E (y^2 + z^2) dxdydz = \int_0^{2\pi} \int_0^1 \int_0^2 r^2 \cdot r dx dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

where we used polar coordinates, $y = r\cos\theta$, $z = r\sin\theta$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

10.

$$RHS = \frac{1}{2}\int_C \mathbf{F}(x,y) \cdot d\mathbf{r} = \frac{1}{2}\int_C -ydx + xdy \underset{Theorem}{\overset{Green's}{=\!=}} \frac{1}{2}\iint_A (1 - (-1)) dxdy$$

$$= \iint_A 1 \; dxdy = Area(A) = LHS$$

11. (a)

$$curl(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & z^2 & 2yz \end{vmatrix} = (2z - 2z)\mathbf{i} - (0-0)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}$$

so the integral of $\mathbf{F}$ along a curve C joining P to Q is independent of the curve C.

(b) By part(a), there exists a function f such that $\mathbf{F} = \nabla f$,

$$f_x = 2x,\ f_y = z^2,\ f_z = 2yz$$

use the method we usually do, we can get $f(x, y, z) = x^2 + yz^2$

$$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = f(1, 3, -1) - f(0, 1, 0) = 1 + 3 - 0 = 4$$

12. (a)

$$\mathbf{r}'(t) = (-2\sin t, 2\cos t, 3), \quad ||\mathbf{r}'(t)|| = \sqrt{4\sin^2 t + 4\cos^2 t + 9} = \sqrt{13}$$

$$L = \int_0^{\pi} ||\mathbf{r}'(t)||dt = \sqrt{13}\pi$$

(b) The projection of this paraboloid on xy-plane is the disk $D = \{(x, y)|x^2 + y^2 \leq 1\}$

$$z_x = 2x, \quad z_y = 2y$$

so the surface area

$$S = \iint_D \sqrt{z_x^2 + z_y^2 + 1}\ dA = \iint_D \sqrt{4x^2 + 4y^2 + 1}\ dxdy = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\ rdrd\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(4r^2 + 1)^{3/2}\Big|_0^1 d\theta = \frac{1}{12}(5^{3/2} - 1) \cdot 2\pi = \frac{\pi}{6}(5^{3/2} - 1)$$

13. (a) $x = \sin\phi\cos\theta,\ y = \sin\phi\sin\theta,\ z = \cos\phi,\ 0 \leq \phi \leq \frac{\pi}{2},\ 0 \leq \theta \leq \frac{\pi}{2}$. We can write

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

(b) The flux of $\mathbf{F}$ across S is,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iint_S \mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)dA$$

we can calculate $\mathbf{r}_\phi \times \mathbf{r}_\theta$ first,

$$\begin{aligned} \mathbf{r}_\phi &= (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) \\ \mathbf{r}_\theta &= (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) \end{aligned}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \sin^2\phi\cos\theta\mathbf{i} - (0 - \sin^2\phi\sin\theta)\mathbf{j} + (\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta)\mathbf{k}$$

$$= \sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + \sin\phi\cos\phi\mathbf{k}$$

the z-component of $\mathbf{r}_\phi \times \mathbf{r}_\theta$ is positive for the range of ϕ, so this is the vector we need, otherwise, take its opposite.

$$\mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = (0, 0, \cos^2\phi) \cdot (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi) = \sin\phi\cos^3\phi$$

$$\iint_S \mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)dA = \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi\cos^3\phi d\phi d\theta = \int_0^{\pi/2} -\frac{1}{4}\cos^4\phi\Big|_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} d\phi = \frac{\pi}{8}$$

14. Same as (9)
15. The boundary C of the hemisphere is given by $x^2 + y^2 + z^2 = 9$ and $z = 0$, i.e. $x^2 + y^2 = 9, z = 0$, so the parametrization of the curve is $\mathbf{r}(t) = (3\cos t,\ 3\sin t,\ 0)$, $0 \le t \le 2\pi$.

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (6\sin t, 0, 3\cos t e^{3\sin t}) \cdot (-3\sin t, 3\cos t, 0) = -18\sin^2 t$$

Use Stokes theorem

$$\int_S curl(\mathbf{F}) \cdot \mathbf{n} dS = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} -18\sin^2 t\ dt = -9\int_0^{2\pi} (1 - \cos 2t)\ dt$$

$$= -9(t - \frac{1}{2}\sin 2t)\Big|_0^{2\pi} = -18\pi$$