Honors Calculus, Sample Final 2.

(1) Given a series $\sum_{K=0}^{\infty} a_K$, define $S_n = \sum_{K=0}^{n} a_K$ be the partial sums of the given series.

- Convergent: we say $\sum_{K=0}^{\infty} a_K$ converges if $\lim_{n\to\infty} S_n$ exists.

  let $a_K = (\frac{1}{2})^K$ then $S_n = \sum_{K=0}^{n} (\frac{1}{2})^K = \frac{1(1-(\frac{1}{2})^{n+1})}{1-\frac{1}{2}} = 2(1-(\frac{1}{2})^{n+1})$

  and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2(1-(\frac{1}{2})^n) = 2$ which converges.

- Divergent; we say $\sum_{K=0}^{\infty} a_K$ diverges if $\lim_{n\to\infty} S_n$ D.N.E.

  let $a_K = (\frac{3}{2})^K$ then $S_n = \sum_{K=0}^{n} (\frac{3}{2})^K = \frac{1[(\frac{3}{2})^{n+1}-1]}{\frac{3}{2}-1} = 2[(\frac{3}{2})^{n+1}-1]$

  and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2[(\frac{3}{2})^{n+1}-1] = \infty$ (D.N.E)

- abs. convergent; we say $\sum_{K=0}^{\infty} a_K$ converges absolutely if $\sum_{K=0}^{\infty} |a_K|$ converges.

  let $a_K = \frac{(-1)^K}{K^2}$. and $\sum_{K=0}^{\infty} |a_K| = \sum_{K=0}^{\infty} \frac{1}{K^2}$ converges (by p-series)

  so $\sum_{K=0}^{\infty} a_K$ converges absolutely.

- conditionally convergent; we say $\sum_{K=0}^{\infty} a_K$ converges conditionally if $\sum_{K=0}^{\infty} a_K$ converges but $\sum_{K=0}^{\infty} |a_K|$ may not converge.

  let $a_K = \frac{(-1)^K}{K}$ and $\sum_{K=0}^{\infty} a_K$ converges by A.S.T. (alternating series test)

  but $\sum_{K=0}^{\infty} |a_K| = \sum_{K=0}^{\infty} \frac{1}{K}$ diverges by p-series test.

(2) (i) "If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges" is false.

example: let $a_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

(ii) "If $\lim_{n\to\infty} n^{\frac{3}{2}} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges" is true.

Since, by the definition of limit, $\lim_{n\to\infty} n^{\frac{3}{2}} a_n = 0$ implies

$|n^{\frac{3}{2}} a_n| < 1$ for a large n, then we have

$0 < |a_n| < \frac{1}{|n^{\frac{3}{2}}|}$. Since $\sum_{n=0}^{\infty} \frac{1}{|n^{\frac{3}{2}}|}$ converges by p-series.

then, by direct comparison test, $\sum_{n=0}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=0}^{\infty} a_n$ conv.

(iii) "If $\lim_{n\to\infty} \sqrt{n}\, a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges" is false.

let $a_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0$ but $\sum_{n=0}^{\infty} \frac{1}{n}$ div.

(3) (a) Given $\sum_{n=1}^{\infty} \frac{\sin(n)}{2^n}$. Consider $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{2^n}\right|$.

Since $0 < \left|\frac{\sin(n)}{2^n}\right| < \frac{1}{2^n}$ and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges by geometric series test $(\frac{1}{2} < 1)$

Then, by comparison series test, $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{2^n}\right|$ converges.

So $\sum_{n=0}^{\infty} \frac{\sin(n)}{2^n}$ converges absolutely.

(3)

(b) Given $\sum_{n=2}^{\infty} \frac{2}{\sqrt{\ln(n)}}$. since $n > \ln(n)$ for $n>0$.

then $\sqrt{n} > \sqrt{\ln(n)}$ and $0 < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{\ln(n)}}$

since $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}}$ diverges by p-series test, then, by comparison test, $\sum_{n=2}^{\infty} \frac{2}{\sqrt{\ln(n)}}$ diverges.

(c) Given $\sum_{n=1}^{\infty} \frac{e^n}{(n!)}$. let $a_n = \frac{e^n}{n!}$. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n}\right| = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$

so, by ratio test, $\sum_{n=1}^{\infty} \frac{e^n}{(n!)}$ converges.

(d) Given $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{2n^2+3n+2}}$ let $b_n = \frac{1}{\sqrt{2n^2+3n+2}}$

since $\lim_{n\to\infty} b_n = 0$ and $b_n > b_{n+1}$, so by A.S.T.

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{2n^2+3n+2}}$ converges.

(4) (i) since $\sin(x) = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$

then $\sin(3x) = \frac{1}{1!}(3x) - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5$

so $T_5 = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!}$

and $R_5 \le \frac{M}{6!}|x|^6$. since $|\sin^{(6)}(3x)| \le 1$. and $|x| < \frac{1}{2}$

$\left(|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |f^{(n+1)}(x)| \le M \text{ as } |x-a| \le d \right)$

then $R_5 \le \frac{1}{2^6 \cdot 6!}$

(4)

(ii) $\sin(x) - T_n(x) = R_n(x) \leq \dfrac{M \cdot |x|^{n+1}}{(n+1)!}$

for fixed $x \in \mathbb{R}$, we have $|\sin^{(n+1)}(x)| \leq 1$ and

$$\frac{M|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as } n \to \infty.$$

So, as $n \to \infty$, $\sin(x) = \lim_{n\to\infty} T_n = \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$

(5)

(i) since $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

and it is convergent for $x \in (-1,1)$

then $\dfrac{2x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n 2 \cdot x^{2n+1}$. $\forall x \in (-1,1)$.

(ii) since $\dfrac{d}{dx}\left(\ln(1+x^2)\right) = \dfrac{2x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$

then $\ln(1+x^2) = \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}\, dx$. $\forall x \in (-1,1)$

$$= \sum_{n=0}^{\infty} (-1)^n \int 2x^{2n+1}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{2x^{2n+2}}{2n+2} + c$$

as $x=0$. $\ln(1+0) = 0 + c \Rightarrow c = 0$.

$$\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{2x^{2n+2}}{2n+2}$$