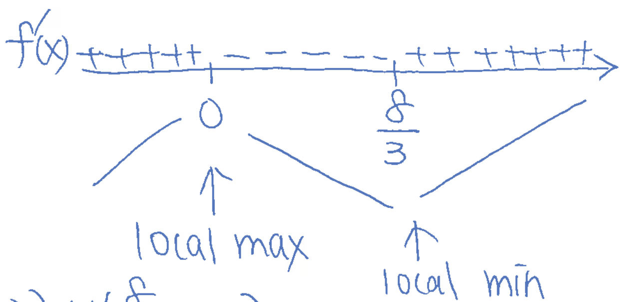
Honors Calculus, Sample First Midterm (b) – Solutions

(1) Given $f(x) = \frac{x^3}{4} - x^2 + 1$ and the domain of f is $(-\infty, \infty)$

Checking the critical point(s): $f'(x) = \frac{3}{4}x^2 - 2x$

$$\Rightarrow \frac{3}{4}x^2 - 2x = 0 \Rightarrow x\left(\frac{3}{4}x - 2\right) = 0 \Rightarrow x = 0 \text{ or } \frac{8}{3}$$

checking the number line.



(a) Increasing interval $(-\infty, 0) \cup (\frac{8}{3}, -\infty)$

(b) decreasing interval $(0, \frac{8}{3})$

(c) local max: $f(0) = 1$; local min $f(\frac{8}{3}) = \frac{1}{4}(\frac{8}{3})^3 - (\frac{8}{3})^2 + 1$

(d) there are no abs. max. and abs. min.

(2) let $f(x) = 2x - 1 - \sin(x)$. To prove $f(x)$ has exactly one root,

We have: ① Contradict proof

Assume $f(x)$ has two roots a and b, $a \neq b$, W.L.O.G. $a < b$.

and $f(a) = 0$, $f(b) = 0$,

Since $f'(x) = 2 - \cos(x) > 0$ $(|\cos(x)| \leq 1)$ f is strictly increasing.

However, by MVT, we have

there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0$$ which is a contradiction

So $f(x)$ has at most one root.

(2) ② Since $f(0)=2\cdot 0-1-\sin(0)=-1<0$. and
$f(\pi)=2\cdot\pi-1-\sin(\pi)=2\pi-1>0$

Then, by Intermedia Value thm, there is a number

$d\in(0,\pi)$. such that $f(0)<0<f(\pi)$ and $f(d)=0$.

So d is the root of f.

(3) Given $f(x)=\ln(x^2+x+1)$ on $[-1,1]$, To find the abs. extreme,

we have.

checking critical number: $f'(x)=\dfrac{2x+1}{x^2+x+1}$

$$\Rightarrow\begin{cases} f'(x)=0 & \Rightarrow 2x+1=0. \quad x=-\frac{1}{2} \\ f'(x)\ \text{DNE} & \Rightarrow x^2+x+1=0 \quad x=\dfrac{-1\pm\sqrt{-3}}{2}\notin\mathbb{R}\ \times \end{cases}$$

check (end) points $f(-\frac{1}{2})=\ln(\frac{3}{4})<0\Rightarrow$ abs. min.

$f(-1)=\ln(1)=0$.

$f(1)=\ln 3>0\Rightarrow$ abs. max.

(4)

(a) $$\lim_{x\to\infty}x^2e^{-\sqrt{x}}\overset{\infty\cdot 0}{=}\lim_{x\to\infty}\frac{x^2}{e^{\sqrt{x}}}\overset{(\frac{\infty}{\infty})}{\underset{L'}{=}}\lim_{x\to\infty}\frac{2x}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}}$$

$$=\lim_{x\to\infty}\frac{4x\sqrt{x}}{e^{\sqrt{x}}}\overset{(\frac{\infty}{\infty})}{\underset{L'}{=}}\lim_{x\to\infty}\frac{6\sqrt{x}}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}}=\lim_{x\to\infty}\frac{12x}{e^{\sqrt{x}}}$$

$$\overset{(\frac{\infty}{\infty})}{\underset{L'}{=}}\lim_{x\to\infty}\frac{12}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}}=\lim_{x\to\infty}\frac{24\sqrt{x}}{e^{\sqrt{x}}}\overset{(\frac{\infty}{\infty})}{\underset{L'}{=}}\lim_{x\to\infty}\frac{12\frac{1}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}}=0.$$

(4) (conti)

(b) $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{\sin(x)} \overset{(\frac{0}{0})}{\underset{L'}{=}} \lim\limits_{x\to 0} \dfrac{-\sin(x)}{\cos(x)} = 0$

(c) $\lim\limits_{x\to\infty} \dfrac{3x^2+2x+1}{\sqrt{2x^4+x^2+2}} = \dfrac{3}{\sqrt{2}}$ (leading coefficient).

(d) $\lim\limits_{x\to 1} |x-1|\cdot \ln|x-1| = \begin{cases} \lim\limits_{x\to 1^+} (x-1)\ln(x-1) \overset{0\cdot\infty}{=} (i) \\ \lim\limits_{x\to 1^-} (1-x)\ln(1-x) = (ii) \end{cases}$

$(i) = \lim\limits_{x\to 1^+} \dfrac{\ln(x-1)}{\frac{1}{x-1}} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim\limits_{x\to 1^+} \dfrac{\frac{1}{x-1}}{-\frac{1}{(x-1)^2}} = \lim\limits_{x\to 1^+} -(x-1) = 0.$

$(ii) = \lim\limits_{x\to 1^-} \dfrac{\ln(1-x)}{\frac{1}{1-x}} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim\limits_{x\to 1^-} \dfrac{\frac{-1}{1-x}}{\frac{1}{(1-x)^2}} = \lim\limits_{x\to 1^-} -(1-x) = 0.$

$\Rightarrow \lim\limits_{x\to 1} |x-1| \ln|x-1| = 0$

(e) $\lim\limits_{x\to 0} \dfrac{\sin(x^7)}{(2x)^7} = \lim\limits_{x^7\to 0} \dfrac{\sin(x^7)}{2^7 x^7} = \dfrac{1}{128} \lim\limits_{x^7\to 0} \dfrac{\sin(x^7)}{x^7} = \dfrac{1}{128}.$

(5). Given $x(t)$ be a position of a particle and $\dot{x}(t)$ is the velocity.

and we have $\dot{x}^2(t) + x^2(t) = c$

Suppose, as $t=0$, $x(0)=0$, $\dot{x}(0)=3$, then $C = 3^2+0^2 = 9$.

$\Rightarrow \dot{x}^2(t) + x^2(t) = 9$

Method 1

let $y(t) = \dot{x}(t)$, we have $y^2(t) + x^2(t) = 9$.

A good guess of $x(t) = 3\sin(t)$, and $y(t) = 3\cos(t)$.

(check: $\dot{x}(t) = (3\sin(t))' = 3\cos(t) = y(t)$ ✓).

So $\ddot{x}(t) = -3\sin(t)$ and the max value of $x(t) = 3\sin(t)$ is 3

the max. value of $\dot{x}(t) = 3\cos(t)$ is 3, and the max value of

$\ddot{x} = -3\sin(t)$ is 3.

Method 2 If we have no luck to guess a suitable solution,

Since, as $\dot{x}(t) = 0$, we have the local extreme of $x(t)$ will be

(and $\dot{x}^2(t) + x^2(t) = 9$

$\Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. since x^2 and $\dot{x}^2$ are always positive.

So if $\dot{x} = 0$. $\dot{x}^2 = 0$. we can get the max of x which $x = 3$.

Do "$\frac{d}{dt}$" on $\dot{x}^2 + x^2 = 9$, we have $2\dot{x}\ddot{x} + 2x \cdot \dot{x} = 0$ and $x = \sqrt{9 - \dot{x}^2}$

As $\ddot{x} = 0$, we have the local extreme of $\dot{x}(t)$:

$$2 \cdot \dot{x} \cdot 0 + 2\left(\sqrt{9 - \dot{x}^2}\right)\dot{x} = 0 \Rightarrow \dot{x} = 0 \text{ or } \dot{x} = \pm 3$$

Similarly, $\dot{x}$ has the max value 3.

Since $2\dot{x}\ddot{x} + 2x \cdot \dot{x} = 0 \Rightarrow x = -\ddot{x}$ and the min value of $x = -3$.

So the max value of $\ddot{x} = 3$.

16) Let $y-2=m(x-1)$ be a line with slope m and pass through point $(1,2)$.

(a) We have the x-intercept $(1-\frac{2}{m},0)$ and y-intercept $(0,2-m)$

So the distance between them is $S=\sqrt{(1-\frac{2}{m})^2+(2-m)^2}$

$$\Rightarrow S^2=(1-\frac{2}{m})^2+(2-m)^2$$

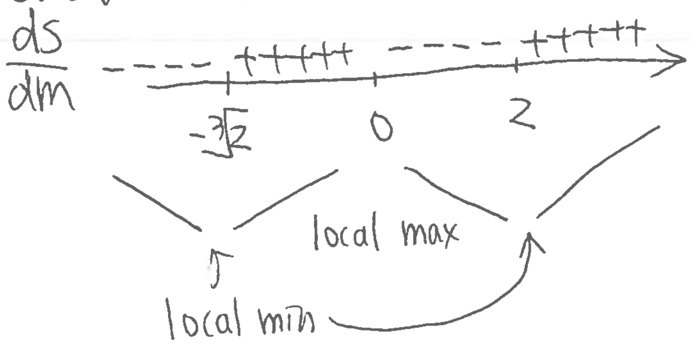
Find the min value of S, we have to check $\frac{dS}{dm}$

$$2S\cdot\frac{dS}{dm}=2(1-\frac{2}{m})\cdot\frac{+2}{m^2}+2(2-m)(-1)$$

$$\frac{dS}{dm}=\frac{\frac{4}{m^2}(1-\frac{2}{m})-2(2-m)}{2S}=\frac{\frac{2}{m^2}(1-\frac{2}{m})-(2-m)}{\sqrt{(1-\frac{2}{m})^2+(2-m)^2}}=\frac{2m-4-2m^3+m^4}{m^3\sqrt{(1-\frac{2}{m})^2+(2-m)^2}}$$

$$\left\{\begin{array}{l}\frac{dS}{dm}=0\Rightarrow m^4-2m^3+2m-4=0\Rightarrow(m^3+2)(m-2)=0\\ \qquad\Rightarrow m=2 \text{ or } -\sqrt[3]{2}.\\ \& \\ \frac{dS}{dm}\text{ DNE}\Rightarrow m^3\sqrt{(1-\frac{2}{m})^2+(2-m)^2}=0\Rightarrow m=0 \text{ or } m=2\end{array}\right.$$

Check the number line:



as $m=2$, $S=0$

as $m=-\sqrt[3]{2}$, $S>0$

(b) Maximum Area in first quadrant: we have

$A=(1-\frac{2}{m})(2-m)\cdot\frac{1}{2}$ and $1-\frac{2}{m}>0$, $2-m>0\Rightarrow m<2$

$$=\frac{(m-2)(2-m)}{2m}=-\frac{(m-2)^2}{2m}$$

So check $\frac{dA}{dm} = \frac{-2(m-2)\cdot 2m + 2(m-2)^2}{4m^2} = \frac{(m-2)[-4m+2m-4]}{4m} = \frac{(m-2)(-2m-4)}{4m}$

and $\frac{dA}{dm} = 0 \Rightarrow m = 2$ or -2

$\frac{dA}{dm}$ DNE $\Rightarrow m = 0$

Check the number line

For m, there is impossible to have a finite max. area. except $A=\infty$, so there is no max. area.