Honors Calculus, Sample ~~First~~ Midterm (a)

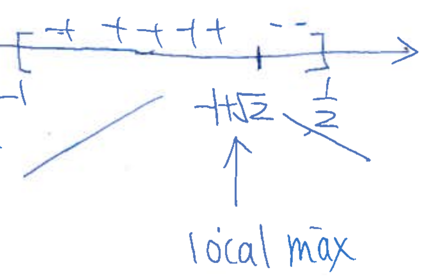
(1) Given $f(x)=\frac{x+1}{x^2+1}$ on $[-1,\frac{1}{2}]$.

Checking $f'(x)=\frac{(x^2+1)-2x(x+1)}{(x^2+1)^2}=\frac{-x^2-2x+1}{(x^2+1)^2}$

$\Rightarrow f'(x)=0 \Rightarrow -x^2-2x+1=0 \Rightarrow x=\frac{2\pm 2\sqrt{2}}{-2}=-1\pm\sqrt{2}$ (NO $-1-\sqrt{2}$ since $-1-\sqrt{2}\notin[-1,\frac{1}{2}]$)

(NO DNE $f'(x)$) $\Rightarrow x=-1+\sqrt{2}$

check the number line: $f'(x)$



$f(-1+\sqrt{2})=\frac{\sqrt{2}}{4-2\sqrt{2}}$ is local max

Check the endpoint(s)

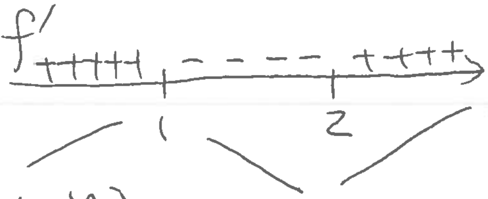
$f(-1)=0$, $f(\frac{1}{2})=\frac{\frac{3}{2}}{\frac{5}{4}}=\frac{3}{2}\times\frac{4}{5}=\frac{6}{5}$ $f(-1+\sqrt{2})=\frac{\sqrt{2}(4+2\sqrt{2})}{8}$

↑ abs min. is abs max.

(2) let $f(x)=2x^3-9x^2+12x+8$. check $f'(x)=6x^2-18x+12$

$\Rightarrow 6x^2-18x+12=0 \Rightarrow x^2-3x+2=0 \Rightarrow (x-1)(x-2)=0 \Rightarrow x=1$ or 2

check the number line f'



$f(1)=13$

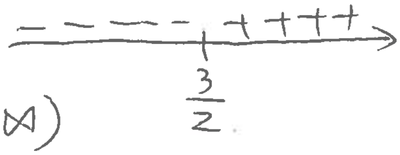
$f(2)=12$.

(a) increasing interval $(-\infty,1)\cup(2,\infty)$

(b) decreasing interval $(1,2)$

check the $f''(x)=12x-18=0 \Rightarrow x=\frac{3}{2}$

check the number line f''



$f(\frac{3}{2})=\frac{27}{4}-\frac{81}{4}+26$

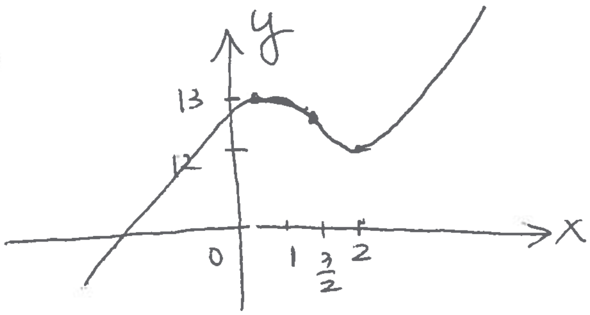
$=\frac{-27}{2}+26=\frac{25}{2}$

(c) concave-up interval : $(\frac{3}{2},\infty)$

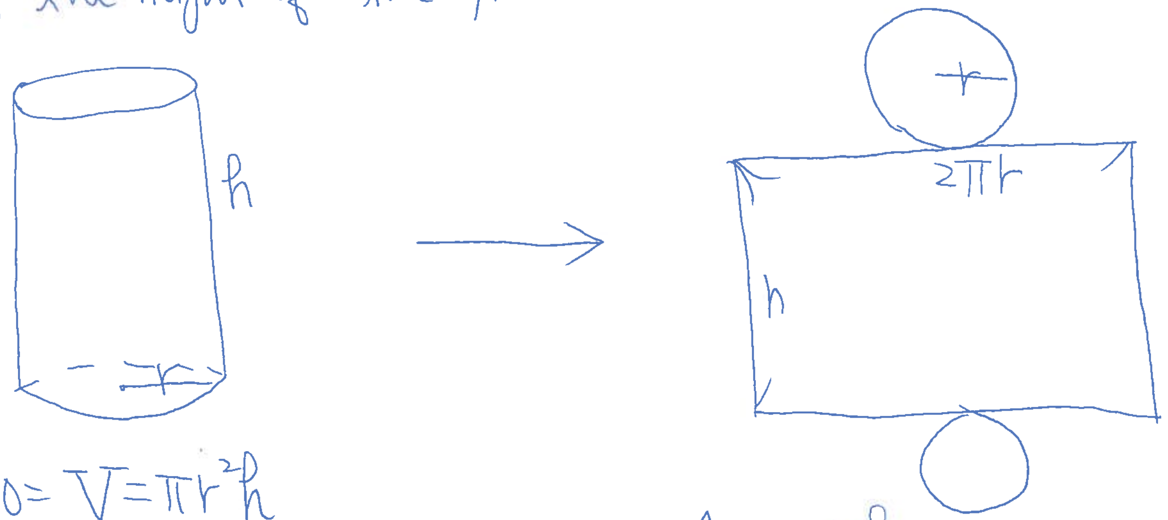
(d) concave-down interval $(-\infty,\frac{3}{2})$.

(2) (conti.)

graph of f:



(3) let the height of the cylinder be h and radius of the circle be r.



$1600 = V = \pi r^2 h$

$A = 2\pi r h + 2r^2\pi$

$\Rightarrow h = \frac{1600}{\pi r^2}$ So $A = 2\pi \cancel{r} \cdot \frac{1600}{\pi r^{\cancel{2}}} + 2r^2\pi = \frac{3200}{r} + 2r^2\pi$

To find the min. value of A, check $\frac{dA}{dr} = \frac{-3200}{r^2} + 4r\pi = \frac{-3200+4r^3\pi}{r^2}$

$\Rightarrow \frac{dA}{dr} = 0 \Rightarrow -3200 + 4r^3\pi = 0 \Rightarrow r^3 = \frac{\overset{800}{\cancel{3200}}}{\cancel{4}\pi} \Rightarrow r = \sqrt[3]{\frac{800}{\pi}}$

$\left\{ \frac{dA}{dr} \text{ DNE} \Rightarrow r = 0 \right.$

check the number line:

So when $r = \sqrt[3]{\frac{800}{\pi}}$, $h = \frac{1600}{\pi \cdot \sqrt[3]{\left(\frac{800}{\pi}\right)^2}}$ A has the min. value.

(4) let U be the mass of uranium. then

(a) $\frac{dU}{dt} = -kU$ where k is positive.

(b) $\therefore\ U = U(0)e^{-kt}$ where $U(0)$ is the initial mass.

We have $U(0)=10$ and $U(2)=8$. then

$$8 = U(2) = 10\cdot e^{-2k} \Rightarrow 0.8 = e^{-2k} \Rightarrow k = \frac{\ln(0.8)}{-2}.$$

So After another 3 years, we have

$$U(5) = 10\cdot e^{-5\cdot\frac{\ln(0.8)}{-2}} = 10\cdot e^{\ln(0.8)^{\frac{5}{2}}} = 10\cdot(0.8)^{\frac{5}{2}}.$$

(5) Suppose f is differentiable on $(-2,6)$, $f(-1)=1$ & $-3 \le f'(x) \le 3$ for all $x \in (-1,2)$.

To show $-5 \le f(1) \le 7$, we have.

By MVT. there is a number $c \in (-1,1)$ such that

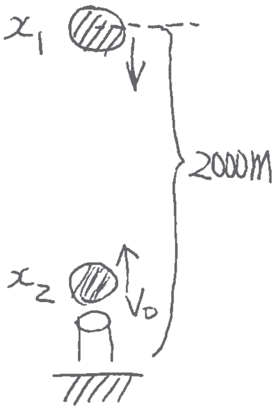
$$f'(c) = \frac{f(1)-f(-1)}{1-(-1)} = \frac{f(1)-1}{2}.$$

Since $-3 \le f'(x) \le 3$ for all $x \in (-1,2) \subset (-1,1)$, so.

$$-3 \le \frac{f(1)-1}{2} \le 3 \Rightarrow -6 \le f(1)-1 \le 6$$

$$\Rightarrow -5 \le f(1) \le 7$$

(6). Given $m\ddot{x}=-mg$ and $g=10\,m/s^2$



(a) we have $\ddot{x}=-g=-10$. and $x_1(0)=2000$

Then $x_1(t)=2000+\frac{1}{2}\cdot g t^2 = 2000-5t^2$

As $x_1(t)=0$, the ball hits the ground. so

$2000-5t^2=0 \Rightarrow t^2=400 \Rightarrow t=20$ (s)

(b) We have $x_2(t)=V_0t-\frac{1}{2}gt^2=V_0t-5t^2$.

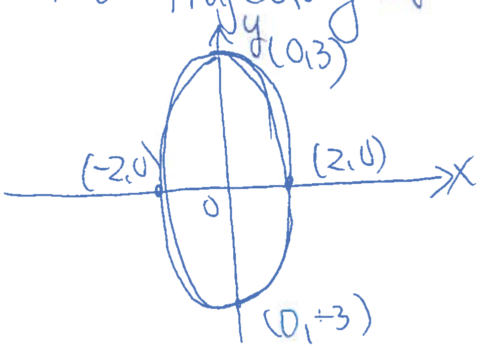
Once two balls hit, we obtain $x_1(t)=x_2(t)$. as $t<20$.

$\Rightarrow 2000-5t^2=V_0t-5t^2 \Rightarrow V_0=\frac{2000}{t} \Rightarrow V_0>\frac{2000}{20}=100$

So the min. V_0 is 100.

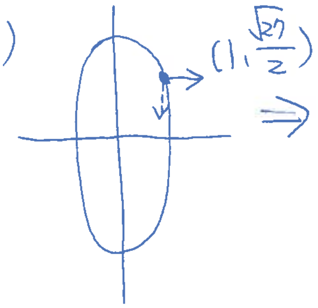
(7) Given a trajectory of a particle $\frac{x^2}{4}+\frac{y^2}{9}=1$

(a)



(b) Given $\frac{dx}{dt}\Big|_{(1,\frac{\sqrt{27}}{2})}=1$

(i)



$\Rightarrow$ clockwise.

(b)(ii)

do $\frac{d}{dt}$ on $\left(\frac{x^2}{4}+\frac{y^2}{9}=1\right)$ we get,

$$\frac{\cancel{2}x}{\cancel{4}2}\cdot\frac{dx}{dt}+\frac{2y}{9}\frac{dy}{dt}=0.$$

$$\Rightarrow \frac{1}{2}\cdot 1+\frac{\sqrt{27}}{9}\cdot\frac{dy}{dt}\Big|_{(1,\frac{\sqrt{27}}{2})}=0$$

$$\Rightarrow \frac{dy}{dt}\Big|_{(1,\frac{\sqrt{27}}{2})}=-\frac{1}{2}\times\frac{9}{\sqrt{27}}=-\frac{\sqrt{3}}{2}$$

(iii) $S^2=x^2+y^2 \xrightarrow{\frac{d}{dt}} 2S\frac{dS}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$

$$\Rightarrow \frac{ds}{dt}\Big|_{(1,\frac{\sqrt{27}}{2})}=\frac{x\frac{dx}{dt}+y\frac{dy}{dt}}{S}\Big|_{(1,\frac{\sqrt{27}}{2})}$$

$$=\frac{1\cdot1+\frac{\sqrt{27}}{2}\left(-\frac{\sqrt{3}}{2}\right)}{\sqrt{1+\frac{27}{4}}}=\frac{2\cdot(2-\frac{9}{4})}{\sqrt{31}}=-\frac{\sqrt{31}}{62}$$