Honors Calculus, Midterm 2 Practice 2 - Solution

(1)

(a) $$\int \frac{\sqrt{2}}{x^2+1}\,dx = \sqrt{2}\arctan(x) + C$$

(b) $$\int \frac{(x-1)^2}{(x-2)^2}\,dx = \int \left(1 - \frac{(x-2)^2-(x-1)^2}{(x-2)^2}\right)dx = \int \left(1+\frac{2x-3}{(x-2)^2}\right)dx$$

$$= \int\left(1+\frac{2x-4+1}{(x-2)^2}\right)dx = \int\left[1+\frac{2}{x-2}+\frac{1}{(x-2)^2}\right]dx$$

$$= x + 2\ln|x-2| - \frac{1}{(x-2)} + C$$

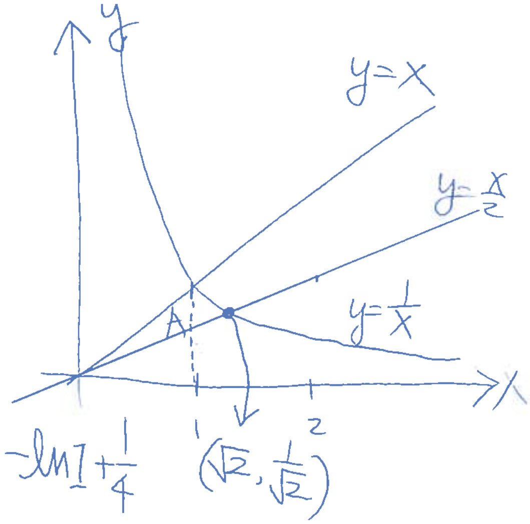
(c) $$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

$$\boxed{\begin{array}{ll} \text{let } u = \ln x & dv = dx \\ du = \frac{dx}{x} & v = x \end{array}}$$

(d) $$\int x^2 e^{3x}\,dx = \frac{x^2}{3}e^{3x} - \frac{2x}{9}e^{3x} + \frac{2}{27}e^{3x} + C.$$

u	dv	sign
x^2	e^{3x}	+
$2x$	$\frac{e^{3x}}{3}$	−
2	$\frac{e^{3x}}{9}$	+
0	$\frac{e^{3x}}{27}$	−
STOP →		+

(2) Given $y=x$, $y=\frac{1}{x}$, $x=0$, and $y=\frac{x}{2}$.



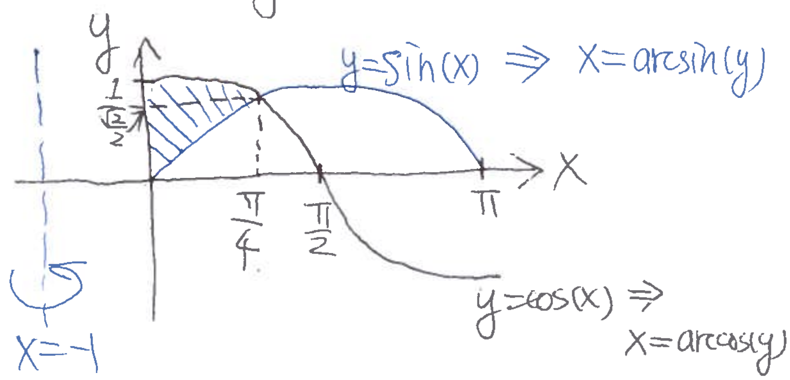
$$A=\int_0^1\left(x-\frac{x}{2}\right)dx+\int_1^{\sqrt{2}}\left(\frac{1}{x}-\frac{x}{2}\right)dx$$

$$=\frac{x^2}{4}\Big|_0^1+\left[\ln|x|-\frac{x^2}{4}\right]_1^{\sqrt{2}}=\frac{1}{4}+\ln(\sqrt{2})-\frac{1}{2}-\ln 1+\frac{1}{4}$$

$$=\ln(\sqrt{2}) \text{ or } \frac{1}{2}\ln 2$$

(3) Given $y=\sin x$, $y=\cos(x)$ and $x=0$, rotating axis is $x=-1$.

(a) Method of cylindrical shell;



$h(x)=\cos(x)-\sin(x)$,

$r(x)=x-(-1)=x+1$

$$V=\int 2\pi r(x)h(x)\,dx=\int_0^{\frac{\pi}{4}}2\pi\cdot\left(\cos(x)-\sin(x)\right)(x+1)\,dx.$$

(b) Method of cross-section:

$$R(y)=\begin{cases}\arcsin(y)+1, & 0\le y\le\frac{\sqrt{2}}{2};\\ \arccos(y)+1, & \frac{\sqrt{2}}{2}\le y\le 1.\end{cases}$$ and $r(y)=1$. Then

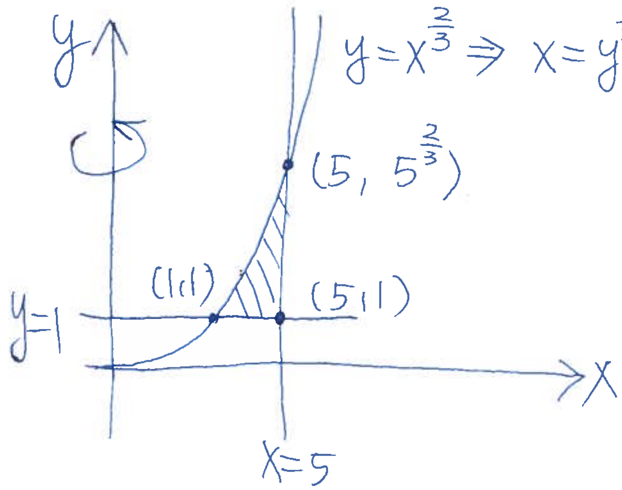
$$V=\pi\int R^2(y)-r^2(y)\,dy=\pi\int_0^{\frac{\sqrt{2}}{2}}(\arcsin(y)+1)^2-1^2\,dy$$

$$+\pi\int_{\frac{\sqrt{2}}{2}}^{1}(\arccos(y)+1)^2-1^2\,dy.$$

(4) Given $y = x^{\frac{2}{3}}$, $x=5$, $y=1$ and rotating axis y axis.

(a) Method of cylindrical shells:

$h(x) = x^{\frac{2}{3}} - 1$, $r(x) = x$



$y = x^{\frac{2}{3}} \Rightarrow x = y^{\frac{3}{2}}$

$$V = 2\pi\int h(x)r(x)\,dx = 2\pi\int_1^5 (x^{\frac{2}{3}}-1)x\,dx$$

$$= 2\pi\int_1^5 (x^{\frac{5}{3}} - x)\,dx = 2\pi\left[\frac{3}{8}x^{\frac{8}{3}} - \frac{x^2}{2}\right]_1^5$$

$$= 2\pi\left[\left(\frac{3}{8}\cdot 5^{\frac{8}{3}} - \frac{25}{2}\right) - \left(\frac{3}{8} - \frac{1}{2}\right)\right] = 2\pi\left(\frac{3}{8}\cdot \underbrace{5^{\frac{8}{3}}}_{5^2\cdot 5^{\frac{2}{3}}} - \frac{99}{8}\right) = \left(\frac{150}{8}\cdot 5^{\frac{2}{3}} - \frac{99}{4}\right)\pi$$

(b) Method of cross-sectional area: $R(y) = 5$, $r(y) = y^{\frac{3}{2}}$

$$V = \pi\int R^2(y) - r^2(y)\,dy = \pi\int_1^{5^{\frac{2}{3}}} 25 - y^3\,dy$$

$$= \pi\left[25y - \frac{y^4}{4}\right]_1^{5^{\frac{2}{3}}} = \pi\left[\left(25\cdot 5^{\frac{2}{3}} - \frac{1}{4}\cdot \underbrace{5^{\frac{8}{3}}}_{5^2\cdot 5^{\frac{2}{3}}}\right) - \left(25 - \frac{1}{4}\right)\right]$$

$$= \pi\left(\frac{75}{4}\cdot 5^{\frac{2}{3}} - \frac{99}{4}\right)$$

(5) Find (a) $\displaystyle\int \frac{1}{(x^2+1)(x-2)}\,dx = \int \frac{-\frac{x}{5} - \frac{2}{5}}{x^2+1} + \frac{\frac{1}{5}}{x-2}\,dx$

$$\frac{Ax+B}{x^2+1} + \frac{\frac{1}{5}}{(x-2)} = \frac{1}{(x^2+1)(x-2)}$$

① As $x=0$, $B - \frac{1}{10} = -\frac{1}{2} \Rightarrow B = -\frac{2}{5}$.

② $Ax+B + \frac{(x^2+1)}{(x-2)}\cdot\frac{1}{5} = \frac{1}{x-2}$

As $x = i \Rightarrow Ai - \frac{2}{5} = \frac{1}{i-2} = \frac{(i+2)}{(i-2)(i+2)} = \frac{i+2}{-5} \Rightarrow A = \frac{-1}{5}$

$$= \int -\frac{1}{5}\frac{x}{x^2+1}\,dx + \int -\frac{2}{5}\frac{1}{x^2+1}\,dx + \int \frac{1}{5}\frac{dx}{x-2}$$

$$= -\frac{1}{5}\cdot\frac{1}{2}\ln(x^2+1) - \frac{2}{5}\arctan(x) + \frac{1}{5}\ln(x-2) + C$$

$$= -\frac{\ln(x^2+1)}{10} - \frac{2}{5}\arctan(x) + \frac{\ln(x-2)}{5} + C.$$

(5)

(b) Since $1 \le x^5 + x^2 + 1 \le 3$ as $0 \le x \le 1$, then

$$\frac{1}{3} \le \frac{1}{x^2 + x^5 + 1} \le 1 \quad \text{as } 0 \le x \le 1$$

$$\Rightarrow \quad \frac{x^2}{3} \le \frac{x^2}{x^5 + x^2 + 1} \le x^2.$$

$$\xRightarrow{\int_0^1 \cdot dx} \quad \frac{1}{9} = \int_0^1 \frac{x^2}{3}\,dx \le \int_0^1 \frac{x^2}{x^5 + x^2 + 1}\,dx \le \int_0^1 x^2\,dx = \frac{1}{3}$$