Honors Calculus, Midterm 2 Sample 3, Solution of (2) (b).

There are ~~three~~ two kinds of forms for arc-length of a given function:

(1) Given $y=f(x)$ on $[a,b]$, the arc-length is

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

(2) Given a parametric function $x(t), y(t)$, $t\in[t_1,t_2]$, the arc-length is $\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$.

~~(3)~~

Now, Given $y=e^x$ on $[0,2]$, so the arc-length of $y=e^x$ on $[0,2]$ is

$$\int_0^2\sqrt{1+e^{2x}}\,dx=\int_0^2\sqrt{1+(e^x)^2}\,dx$$

$\boxed{\begin{aligned}&\text{let } e^x=\tan(\theta)\\ &e^x dx=\sec^2(\theta)d\theta \Rightarrow dx=\frac{\sec^2(\theta)}{e^x}d\theta=\frac{\sec^2(\theta)}{\tan(\theta)}d\theta\end{aligned}}$

the indefinite integral will be

$$\int\sqrt{1+\tan^2(\theta)}\cdot\frac{\sec^2(\theta)}{\tan(\theta)}d\theta=\int\frac{\sec^3(\theta)}{\tan(\theta)}d\theta=\int\frac{\sec(\theta)(\sec^2(\theta))}{\tan(\theta)}d\theta$$

$$=\int\frac{\sec(\theta)(1+\tan^2(\theta))}{\tan(\theta)}d\theta=\int\left(\frac{\sec(\theta)}{\tan(\theta)}+\sec(\theta)\tan(\theta)\right)d\theta$$

$$=\int\frac{1}{\cos(\theta)}\frac{\cos(\theta)}{\sin(\theta)}d\theta+\int\sec(\theta)\tan(\theta)\,d\theta$$

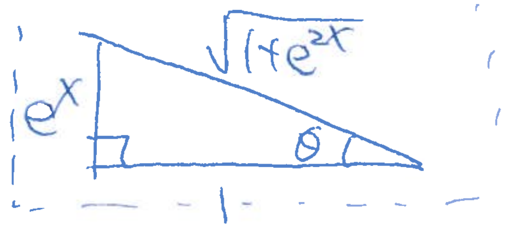
$= \int \frac{d\theta}{\sin(\theta)} + \sec(\theta)$ $\quad$ (since $\int \sec(\theta)\tan(\theta)\,d\theta = \sec(\theta) + C$)

$$= \int \csc(\theta)\,d\theta + \sec(\theta) = \ln|\csc(\theta) - \cot(\theta)| + \sec(\theta) + C.$$

$$\boxed{\int \csc(\theta)\,d\theta = \ln|\csc(\theta) - \cot(\theta)| + C}$$

Change θ back to x. Since $\frac{e^x}{1} = \tan(\theta)$, we have

$\sec(\theta) = \sqrt{1+e^{2x}}$, $\csc(\theta) = \frac{\sqrt{1+e^{2x}}}{e^x}$, $\cot(\theta) = \frac{1}{e^x}$



so $\int \sqrt{1+e^{2x}}\,dx = \ln\left|\frac{\sqrt{1+e^{2x}} - 1}{e^x}\right| + \sqrt{1+e^{2x}} + C.$

then $\int_0^2 \sqrt{1+e^{2x}}\,dx = \ln\left|\frac{\sqrt{1+e^4} - 1}{e^2}\right| - \ln\left|\frac{\sqrt{1+e^0} - 1}{e^0}\right| + \sqrt{1+e^4} - \sqrt{1+e^0}$

$$= \ln\left|\frac{\sqrt{1+e^4} - 1}{e^2}\right| - \ln|\sqrt{2} - 1| + \sqrt{1+e^4} - \sqrt{2}.$$