Honors Calculus, Midterm 2 Sample Solution

(1)

(a) $\int \frac{\pi}{x^2+1}\,dx = \pi \arctan x + C$

(b) $\int \frac{x^2}{(x-4)^2}\,dx \underset{\uparrow}{=} \int \left(\frac{u+4}{u}\right)^2 du = \int \left(1+\frac{4}{u}\right)^2 du$

$\boxed{u = x-4,\ du = dx}$

$= \int 1 + \frac{8}{u} + \frac{16}{u^2}\,du = u + 8\ln|u| - \frac{16}{u} + C$

$= (x-4) + 8\ln|x-4| - \frac{16}{x-4} + C$

Another way:

$$\int \frac{x^2}{(x-4)^2}\,dx = \int \frac{x^2}{x^2-8x+16}\,dx = \int 1 - \frac{-8x+16}{x^2-8x+16}\,dx = \int 1 + \frac{8x-16}{(x-4)^2}\,dx$$

$$= \int 1 + \frac{8}{x-4} + \frac{16}{(x-4)^2}\,dx = x + 8\ln|x-4| - 16\left(\frac{1}{x-4}\right) + C.$$

Find A, B, such that $\frac{A}{x-4} + \frac{B}{(x-4)^2} = \frac{8x-16}{(x-4)^2} = \frac{8x-32-16+32}{(x-4)^2}$

$= \frac{8(x-4)+16}{(x-4)^2} \Rightarrow A=8,\ B=16$

Think about it: Did we get same answer by two different ways?

(c) $\int (\ln\theta)^2\,d\theta \underset{\uparrow}{=} \theta(\ln\theta)^2 - \int 2\ln\theta\,d\theta \underset{\uparrow}{=} \theta(\ln\theta)^2 - 2\theta(\ln\theta) + 2\theta + C.$

let $u = (\ln\theta)^2$, $dv = d\theta$

$du = 2(\ln\theta)\frac{d\theta}{\theta}$, $v = \theta$

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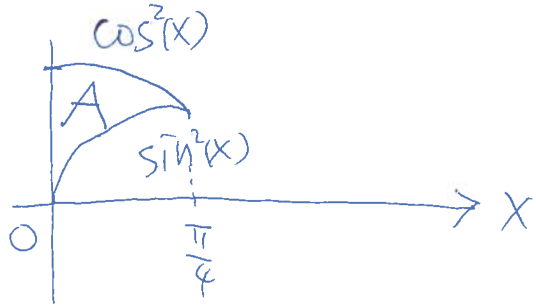
(d) $\int \underset{E}{e^{3x}} \underset{T}{\cos(x)}\,dx = \frac{e^{3x}}{3}\cos(x) + \frac{e^{3x}}{9}\sin(x) - \int \frac{e^{3x}}{9}\cos(x)\,dx$

u	dv	sign
$\cos(x)$	e^{3x}	$+$
$-\sin(x)$	$\frac{e^{3x}}{3}$	$-$
$-\cos(x)$	$\frac{e^{3x}}{9}$	$+$
		$-$

$$\Rightarrow \frac{10}{9}\int e^{3x}\cos(x)\,dx = \frac{e^{3x}}{3}\cos(x) + \frac{e^{3x}}{9}\sin(x) + C'$$

$$\Rightarrow \int e^{3x}\cos(x)\,dx = \frac{3}{10}e^{3x}\cos(x) + \frac{e^{3x}}{10}\sin(x) + C.$$



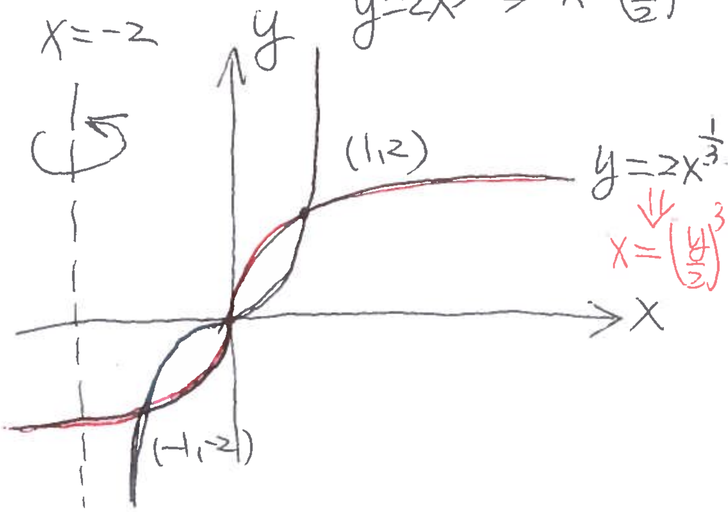
(2) Area of $A = \int_0^{\frac{\pi}{4}} \cos^2(x) - \sin^2(x)\,dx$

$\cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x)$

$$= \int_0^{\frac{\pi}{4}} \cos(2x)\,dx = \frac{1}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}} \quad \left(\cos^2(x) \geq \sin^2(x) \text{ as } 0 \leq x \leq \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\cdot\sin\left(\frac{\pi}{2}\right) - 0 = \frac{1}{2}$$

$y = 2x^3 \Rightarrow x = \left(\frac{y}{2}\right)^{\frac{1}{3}}$



(3)

(a) By cylindrical shells, we have

$$h(x) = \begin{cases} 2x^{\frac{1}{3}} - 2x^3 & , 0 \leq x \leq 1; \\ 2x^3 - 2x^{\frac{1}{3}} & , -1 \leq x \leq 0. \end{cases}$$

and $r(x) = x - (-2) = x + 2$

$$V = 2\pi\int_{-1}^{1} r(x)h(x)\,dx = 2\pi\int_0^1 (2x^{\frac{1}{3}} - 2x^3)(x+2)\,dx$$

$$+ 2\pi\int_{-1}^{0} (2x^3 - 2x^{\frac{1}{3}})(x+2)\,dx.$$

(3)

(b) By method of cross-section, we have

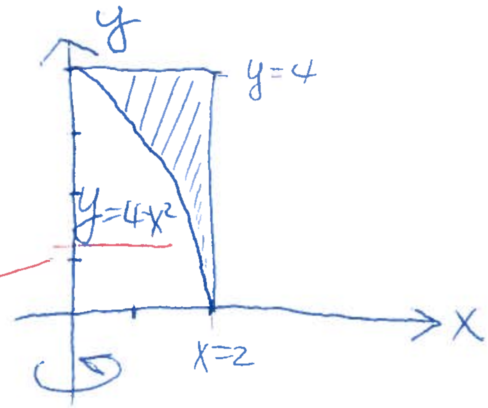
$$V = \pi\int_0^2 \left[\left(\frac{y}{2}\right)^{\frac{1}{3}} - (-2)\right]^2 - \left[\left(\frac{y}{2}\right)^3 - (-2)\right]^2 dy +$$

$$\pi\int_{-2}^0 \left[\left(\frac{y}{2}\right)^3 - (-2)\right]^2 - \left[\left(\frac{y}{2}\right)^{\frac{1}{3}} - (-2)\right]^2 dy.$$

(4) Given $y = 4 - x^2$, $x = 2$, $y = 4$ about y-axis.

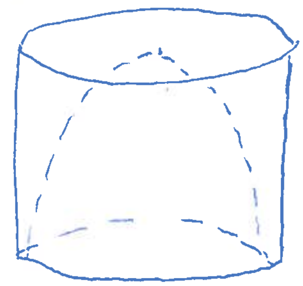
(a) By method of cylindrical shells,

We have $h(x) = 4 - (4 - x^2) = x^2$, $r(x) = x$



$$V = \int 2\pi\, h(x)\, r(x)\, dx \qquad x = \sqrt{4-y}$$

$$= \int_0^2 2\pi \cdot x^3\, dx = 2\pi \frac{x^4}{4}\Big|_0^2 = 2\pi\left(\frac{16}{4} - 0\right) = 4 \cdot 2\pi = 8\pi$$



(b) By method of cross-sectional area,

We have $R(y) = 2$, $r(y) = \sqrt{4-y}$.

$$V = \pi\int_0^4 2^2 - (\sqrt{4-y})^2 dy = \pi\int_0^4 4 - (4-y)\,dy$$

$$= \pi\int_0^4 y\,dy = \pi \frac{y^2}{2}\Big|_0^4 = 8\pi.$$