Honors Calculus, Midterm 2, Sample 2, Solution

(1)

(a) $\int_0^1 x^2\sqrt{1-x^3}\,dx = \int_1^0 -\frac{1}{3}\sqrt{u}\,du = \frac{1}{3}\int_0^1 \sqrt{u}\,du = \frac{2}{9}u^{\frac{3}{2}}\Big|_0^1 = \frac{2}{9}.$

$u = 1-x^3$
$du = -3x^2dx \Rightarrow \frac{du}{-3} = x^2dx$

(b) $\int \frac{dx}{\sqrt{1+x}} = \frac{2}{3}(1+x)^{\frac{3}{2}} + c$

(c) $\int \frac{dx}{x(\ln x)} = \int \frac{du}{u} = \ln|u| + c = \ln|\ln(x)| + c$

$u = \ln(x)$
$du = \frac{dx}{x}$

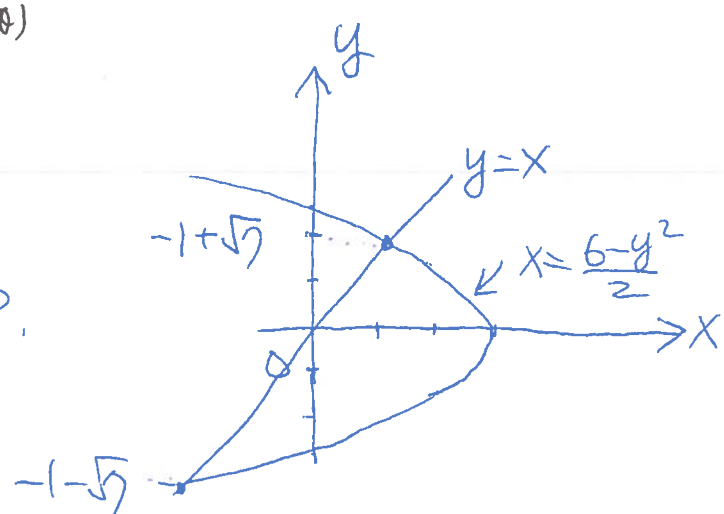
(d) $\int \frac{dx}{x^2+9} = \int \frac{3\sec^2(\theta)}{9\sec^2(\theta)}d\theta = \frac{1}{3}\int d\theta = \frac{1}{3}\theta + c = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$

$x = 3\tan(\theta)$ $\quad x^2+9 = 9\sec^2(\theta)$
$dx = 3\sec^2(\theta)d\theta$

(2) Given $y = x$ and $y^2+2x = 6$

intersection point(s): $y^2+2y-6=0$.

$y = \frac{-2\pm 2\sqrt{7}}{2} = -1\pm\sqrt{7}$



$$\int_{-1-\sqrt{7}}^{-1+\sqrt{7}} \frac{6-y^2}{2} - y\,dy = \frac{1}{2}\int_{-1-\sqrt{7}}^{-1+\sqrt{7}} (6-y^2-2y)\,dy = \frac{1}{2}\left[6y - \frac{y^3}{3} - y^2\right]_{-1-\sqrt{7}}^{-1+\sqrt{7}}$$

$$= \frac{1}{2}\left[6(2\sqrt{7}) - \frac{1}{3}(20\sqrt{7}) - (-4\sqrt{7})\right] = \frac{1}{2}\left[\frac{28}{3}\sqrt{7}\right] = \frac{14}{3}\sqrt{7}$$

(3) $$\int_0^1 \ln(x)\,dx = \lim_{a\to 0}\int_a^1 \ln(x)\,dx = \lim_{a\to 0}\left\{x\ln(x)\Big|_a^1 - \int_a^1 dx\right\}$$

$$\boxed{\begin{array}{ll} u=\ln(x) & dv=dx \\ du=\frac{dx}{x} & v=x \end{array}} \Rightarrow$$

$$= \lim_{a\to 0}\left[\cancel{1\cdot\ln(1)}^{\,0} - a\ln(a) - (1-a)\right] = -\lim_{a\to 0} a\ln(a) - 1 \overset{(0\cdot\infty)}{=} 0 - 1 = -1$$

$$\lim_{a\to 0}\frac{\ln(a)}{\frac{1}{a}} \overset{L'}{=} \lim_{a\to 0}\frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0} -a = 0$$

$$\int_0^2 \frac{dx}{|x-1|^{\frac{1}{2}}} = \int_0^1 \frac{dx}{(1-x)^{\frac{1}{2}}} + \int_1^2 \frac{dx}{(x-1)^{\frac{1}{2}}}$$

$$= \lim_{a\to 1}\int_0^a \frac{dx}{(1-x)^{\frac{1}{2}}} + \lim_{a\to 1}\int_a^2 \frac{dx}{(x-1)^{\frac{1}{2}}}$$

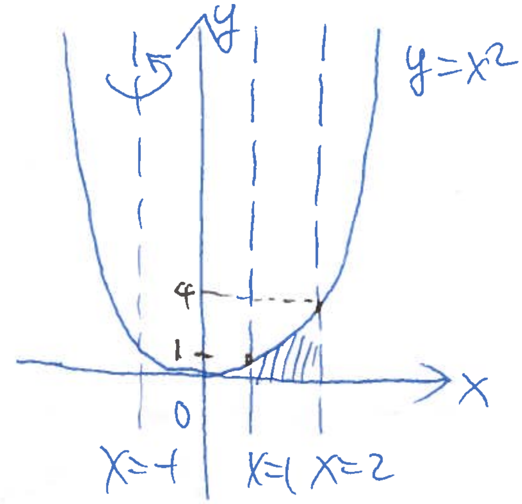
$$= \lim_{a\to 1}\left(-2(1-x)^{\frac{1}{2}}\Big|_0^a + 2(x-1)^{\frac{1}{2}}\Big|_a^2\right)$$

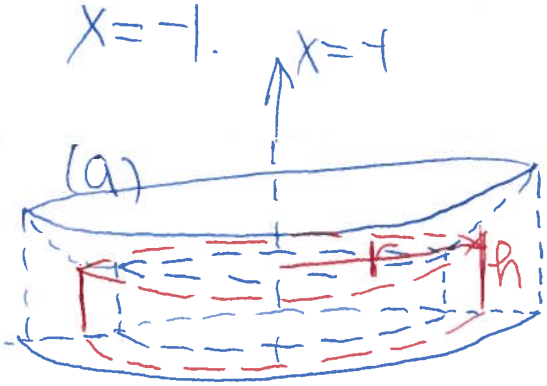
$$= \lim_{a\to 1}\left(-2(1-a)^{\frac{1}{2}} + 2 + 2 - 2(a-1)^{\frac{1}{2}}\right) = 4$$

(4)

Given $y=x^2$, $x=1$, $x=2$, $y=0$.

Find the rotating volume about $x=-1$.





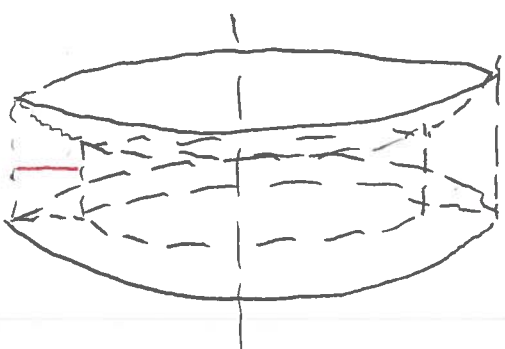
$r = x-(-1)$

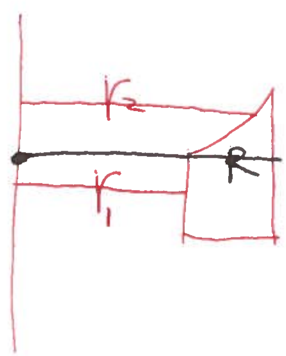
$h = x^2$. $\quad x\in[1,2]$

$$V_R = 2\pi\int_1^2 (x+1)x^2\,dx = 2\pi\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_1^2 = 2\pi\left[\frac{15}{4}+\frac{7}{3}\right]$$

$$= \frac{73}{6}\pi$$

(b)





$R = 2-(-1)=3$

$r_1 = 1-(-1)$ as $y\in[0,1]$

$r_2 = \sqrt{y}-(-1)$ as $y\in[1,4]$

$$V_R = \pi\left[\int_0^1 3^2-2^2\,dy + \int_1^4 \underbrace{3^2-(\sqrt{y}+1)^2}_{9-(y+2\sqrt{y}+1)}\,dy\right]$$

$$= \pi\left[5+\left(8y-\frac{y^2}{2}-\frac{4}{3}y^{\frac{3}{2}}\right)\Big|_1^4\right] = \pi\left[5+\left(24-\frac{15}{2}-\frac{28}{3}\right)\right]$$

$$= \left(29-\frac{101}{6}\right)\pi = \frac{73}{6}\pi$$

(5) (a) $$\int \frac{x}{(x-1)(x+1)(x+2)}\,dx = \int \left(\frac{\frac{1}{6}}{x-1} + \frac{+\frac{1}{2}}{x+1} + \frac{-\frac{2}{3}}{x+2} \right) dx$$

$$= \frac{1}{6}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{2}{3}\ln|x+2| + c$$

(b) since $\sqrt{x^4+1} \geq \sqrt{x^2}$ as $x \in [0,1]$, we have

$$\sqrt{x^4+1} \geq x \Rightarrow \int_0^1 \sqrt{x^4+1}\,dx \geq \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

$$\Rightarrow \int_0^1 \sqrt{x^4+1}\,dx \geq \frac{1}{2}.$$