Honors Calculus, Midterm 2 Sample I Solution

(1)

(a) $\displaystyle\int_0^1 2x\sqrt{1-x^4}\,dx \underset{\substack{\uparrow\\ u=x^2}}{=} \int_0^1 \sqrt{1-u^2}\,du \underset{\uparrow}{=} \boxed{\frac{\pi}{4}}$ or

a quarter of a unit circle

Find the anti-derivative: $\displaystyle\int 2x\sqrt{1-x^4}\,dx \underset{\uparrow}{=} \int \sqrt{1-u^2}\,du \underset{\uparrow}{=} \int \cos^2(\theta)\,d\theta$

by using u-sub, and tri-sub.

$\boxed{\begin{array}{l}\text{let } u=x^2\\ du=2x\,dx\end{array}}$ $\boxed{\begin{array}{l}u=\sin(\theta)\\ du=\cos(\theta)\,d\theta\end{array}}$

$u=\sin(\theta)$

$$= \int \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C = \frac{\arcsin(u)}{2} + u\sqrt{1-u^2} + C$$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$= \frac{\arcsin(x^2)}{2} + x^2\sqrt{1-x^4} + C$$

$$\Rightarrow \int_0^1 2x\sqrt{1-x^4}\,dx = \left.\frac{\arcsin(x^2)}{2} + x^2\sqrt{1-x^4}\right|_0^1 = \frac{1}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{4}}$$

(b) $\displaystyle\int \frac{x^3}{\sqrt{1-x}}\,dx = 2x^3(1-x)^{\frac{1}{2}} - 4x^2(1-x)^{\frac{3}{2}} + \frac{16}{5}(1-x)^{\frac{5}{2}} - \frac{32}{35}(1-x)^{\frac{7}{2}} + C$

u	dv	sign
x^3	$\frac{1}{\sqrt{1-x}}$	+
$3x^2$	$2(1-x)^{\frac{1}{2}}$	−
$6x$	$\frac{4}{3}(1-x)^{\frac{3}{2}}$	+
6	$\frac{8}{15}(1-x)^{\frac{5}{2}}$	−
0	$\frac{16}{105}(1-x)^{\frac{7}{2}}$	+
		−

(c) $\displaystyle\int \frac{dx}{x(\ln x)^2} \underset{\uparrow}{=} \int \frac{du}{u^2} = -\frac{1}{u} + C$

$\boxed{\begin{array}{l}u=\ln x\\ du=\frac{dx}{x}\end{array}}$

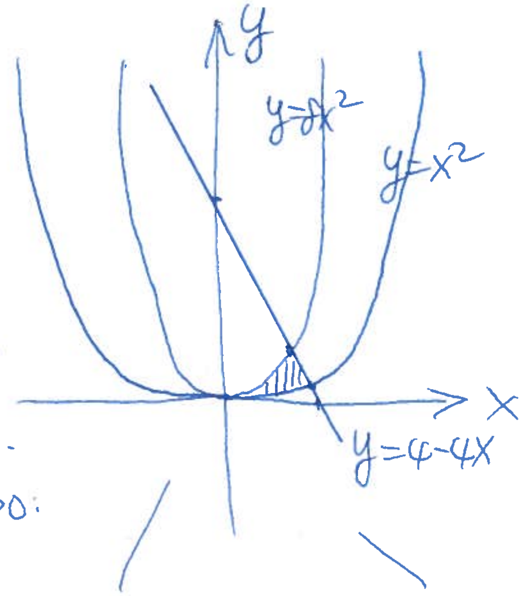
$$= -\frac{1}{\ln(x)} + C.$$

(d) $\displaystyle\int \frac{dx}{2x^2+4} = \frac{1}{2}\int \frac{dx}{x^2+2} \underset{\uparrow}{=} \frac{1}{2}\int \frac{\sqrt{2}\sec^2(\theta)}{2\sec^2(\theta)}\,d\theta = \frac{\sqrt{2}}{4}\theta + C$

$\boxed{\begin{array}{l}x=\sqrt{2}\tan(\theta)\\ dx=\sqrt{2}\sec^2(\theta)\,d\theta\\ x^2+2=2\sec^2\theta\end{array}}$

$$\underset{\substack{\uparrow\\ \theta=\arctan\left(\frac{x}{\sqrt{2}}\right)}}{=} \frac{\sqrt{2}}{4}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.$$

(2) Given $y=x^2$, $y=8x^2$ and $y=4-4x$.



the intersection of $y=x^2$ and $y=4-4x$:

as $x, y>0$.

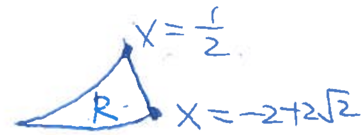
$$x^2+4x-4=0 \Rightarrow x=\frac{-4\pm\sqrt{32}}{2} = -2\pm 2\sqrt{2}$$

$$\Rightarrow x=-2+2\sqrt{2}.$$

the intersection point of $y=8x^2$ and $y=4-4x$.

as $x, y>0$:

$$8x^2+4x-4=0 \Rightarrow x=\frac{1}{2} \text{ or } \times$$



$$A_R = \int_0^{\frac{1}{2}} 8x^2-x^2\,dx + \int_{\frac{1}{2}}^{-2+2\sqrt{2}} (4-4x-x^2)\,dx$$

$$= \frac{7}{3}x^3 \Big|_0^{\frac{1}{2}} + \left[4x-2x^2-\frac{x^3}{3}\right]_{\frac{1}{2}}^{-2+2\sqrt{2}}$$

$$= \frac{7}{24} + \left[4\left(-\frac{5}{2}+2\sqrt{2}\right) - 2\left(12-8\sqrt{2}-\frac{1}{4}\right) - \frac{1}{3}\left(-56+40\sqrt{2}\right)\right]$$

$$= \frac{7}{24} + \left[\underbrace{-10-\frac{27}{2}+\frac{56}{3}}_{-\frac{29}{6}} + 8\sqrt{2} + 16\sqrt{2} - \frac{40}{3}\sqrt{2}\right] = -\frac{109}{24} + \frac{34}{3}\sqrt{2}.$$

(3) $$\int_0^1 \frac{dx}{x^2} = \lim_{a\to 0}\int_a^1 \frac{dx}{x^2} = \lim_{a\to 0}\left[-\frac{1}{x}\right]_a^1 = \lim_{a\to 0}\left[\frac{1}{a}-1\right] \text{ diverges.}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_0^{\infty}\frac{dx}{1+x^2} + \int_{-\infty}^{0}\frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} + \lim_{a\to -\infty}\int_a^0 \frac{dx}{1+x^2}$$

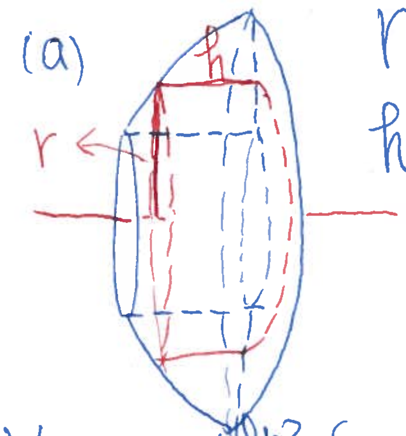
$$= \lim_{b\to\infty}\left[\arctan(x)\right]_0^b + \lim_{a\to -\infty}\left[\arctan(x)\right]_a^0$$

$$= \lim_{b\to\infty}\left[\arctan(b) - \underbrace{\arctan(0)}_{0}\right] + \lim_{a\to -\infty}\left[\underbrace{\arctan(0)}_{0} - \arctan(a)\right]$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

(4) Given $y = \ln x$, $x = 2$, $x = 1$, and $y = 0$ find the rotating volume about $y = -1$

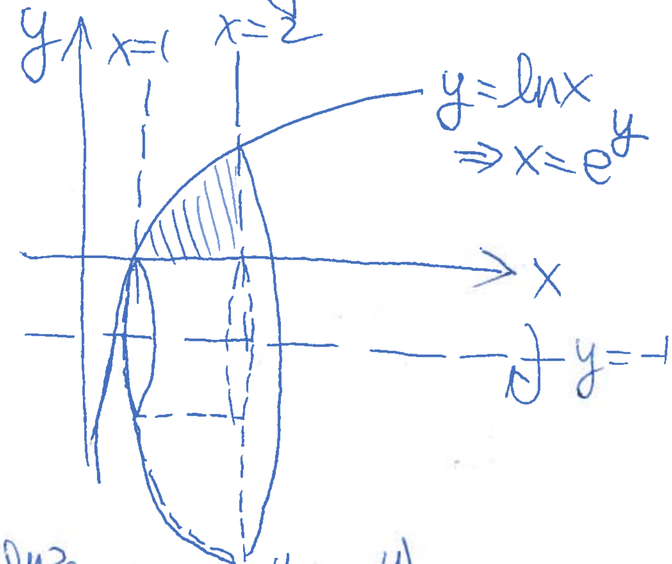
(a)



$r = y - (-1)$

$h = 2 - e^y$

$y \in (0, \ln 2)$

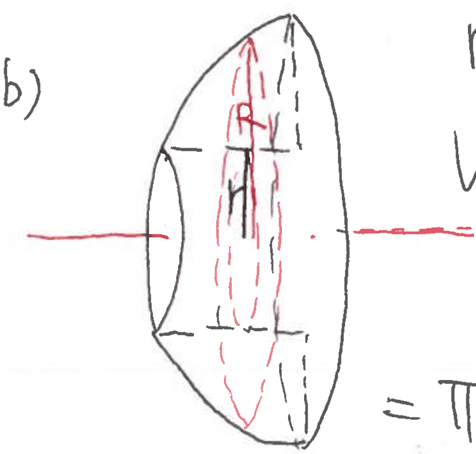


$$V_R = 2\pi \int_0^{\ln 2} (y+1)(2-e^y)\,dy = 2\pi \int_0^{\ln 2} (2 + 2y - e^y - ye^y)\,dy$$

$$= 2\pi \left[2y + y^2 \cancel{- e^y} - ye^y \cancel{+ e^y}\right]_0^{\ln 2} = 2\pi\left[2\ln 2 + (\ln 2)^2 - 2\ln 2\right]$$

$$= 2\pi (\ln 2)^2$$

(b)



$r = 1$, $R = \ln x - (-1)$ $\quad x \in [1, 2]$

$$V_R = \pi \int_1^2 (\ln x + 1)^2 - 1^2\,dx$$

$$= \pi \int_1^2 \left((\ln x)^2 + 2\ln(x) + 1 - 1\right) dx$$

$$= \pi \left[\int_1^2 (\ln x)^2\,dx + \int_1^2 2\ln(x)\,dx\right]$$

$u = (\ln(x))^2$ $\quad dv = dx$

$du = \dfrac{2\ln(x)}{x}$ $\quad v = x$

$$= \pi \left\{ \left[x(\ln(x))^2\right]_1^2 - 2\int_1^2 \ln(x)\,dx + 2\int_1^2 \ln(x)\,dx \right\}$$

$$= \pi \left[2(\ln(2))^2\right]$$

(5)

(a) $\displaystyle\int \frac{dx}{(x-1)(x+1)(x+2)} = \int \left(\frac{\frac{1}{6}}{x-1} + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{3}}{x+2}\right) dx$

$\displaystyle = \frac{1}{6}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{3}|x+2| + C.$

(b) Since $\frac{1}{x}$ is a decreasing function. so if we use Riemann Sum to approach $\int_1^{n+1} \frac{1}{x}dx$ by using right end point of each subinterval $[k, k+1]$ for $k<n$.

We get $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}$ < the area under $\frac{1}{x}$ for x from 1 to $n+1$

$\displaystyle = \int_1^{n+1} \frac{dx}{x}.$