Honors Calculus, Sample Final 4

(1) (a) By Root test, given $\sum_{n=0}^{\infty} a_n (x-a)^n$. we have.

$$\lim_{n\to\infty} \left| \sqrt[n]{a_n}\,(x-a) \right| \underset{\uparrow}{=} |x-a|<1. \Rightarrow -1 < x-a < 1$$

$$\Rightarrow -1+a < x < 1+a$$

if a_n is finite. $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

$\Rightarrow$ convergent interval: $[a-1, a+1]$ and $R=1$.

(b) Given $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. let $a_n = \frac{x^n}{n^2}$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)^2} \frac{n^2}{|x|^n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}|x| = |x|<1.$$

and as $x=1$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series.

as $x=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by A.S.T.

Then the convergent interval of $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is $[-1,1]$

(b) Since $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$. we have. f is differentiable on $(a-R, a+R)$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

then $f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$

$$f''(x) = 2a_2 + 3!\,a_3(x-a) + 4\cdot 3 a_4 (x-a)^2 + \cdots$$

$$f^{(n)}(x) = n!\,a_n + (n+1)!\,a_{n+1}(x-a) + \frac{(n+2)!}{2} a_{n+2}(x-a)^2 + \cdots$$

$$\Rightarrow f^{(n)}(a) = n!\,a_n \Rightarrow a_n = \frac{f^{(n)}(a)}{n!}$$

(c) (i) since $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

then $T_2 = 1 - \frac{x^2}{2!}$

(ii) $R_2(x) \leq \frac{M|x|^3}{3!}$ and $|\cos^{(3)}(x)| \leq 1$ and $|x| < \frac{1}{5}$,

So $R_2(x) \leq \frac{1}{3!} \cdot \frac{1}{5^3} = \frac{1}{5^3 \cdot 3!}$

(2) (a) Given $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$. and $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges by A.S.T

but $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges by p-series test,

So $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

(b) Given $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. since $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln(x)} \Big|_2^{\infty}$ converges.

So, by integral test $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

(c) (i) this is ture by comparison test,

(iii) this is false. let $a_n = \frac{1}{n}$.

$\sum_{n=0}^{\infty} a_n^2 = \sum_{n=0}^{\infty} \frac{1}{n^2}$ converges by p-series test

but $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

(3) (a) Given $\sum_{n=1}^{\infty} \frac{n^3+2n+1}{n^4+n+2}$. let $a_n = \frac{n^3+2n+1}{n^4+n+2}$ and $b_n = \frac{1}{n}$.

and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+2n+1}{n^4+n+2} \cdot n = 1 > 0$ & $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

So, by limit Comparison test, $\sum_{n=1}^{\infty} \frac{n^3+2n+1}{n^4+n+2}$ diverges

(b) Given $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$. let $a_n = \frac{1}{\ln(n)}$.

since $\lim_{n\to\infty} a_n = 0$ and $a_n > a_{n+1} > 0$, So, by A.S.T.

$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges.

(c) Given $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$, let $a_n = \frac{n^2}{e^n}$. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}\right|$

$= \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2}\right| \cdot \frac{1}{e} = \frac{1}{e} < 1$.

So, by ratio test, $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ converges.

(d) Given $\sum_{n=0}^{\infty} \frac{n\sqrt{2}}{(1.2)^n}$. since $0 < \frac{n\sqrt{2}}{(1.2)^n} < \frac{2}{(1.2)^n}$ and $\sum_{n=0}^{\infty} \frac{2}{(1.2)^n}$ converges by geometric series test.

then $\sum_{n=0}^{\infty} \frac{n\sqrt{2}}{(1.2)^n}$ converges by comparison test,

(4)

(i) since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. then $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$

and $x^2 e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+2}}{n!}$

So $T_5 = x^2 + \frac{2x^3}{1!} + \frac{4x^4}{2!} + \frac{8x^5}{3!}$

(ii)

let $f(x) = x^2 e^{2x}$. $f^{(5)}(0) = \frac{8}{3!} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 160$.

(5) (i) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x)^{2n}$ $x \in (-1,1)$

(as $x=1$ & $x=-1$ $\sum_{n=0}^{\infty} (-x^2)^n$ diverges).

(ii) Since $\frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$ $\forall x \in (-1,1)$

$\arctan(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$.

as $x=0$. $\arctan(0) = 0 + C \Rightarrow C = 0$.

So $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $\forall x \in (-1,1)$.