Honors Calculus, Sample Final 4.

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ATTEMPT ALL QUESTIONS. SHOW ALL WORKING. POINTS WILL NOT BE AWARDED IF WORKING IS NOT SHOWN. NO PROGRAMMABLE CALCULATORS ARE TO BE USED. TIME ALLOWED: 90 MINUTES

Please write your answers clearly and in a logical and well-organized way. Points will be deducted for sloppy work.

GOOD LUCK!

(1) [20] (a) [5] Suppose

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

is a power series. In half a page or less describe the radius of convergence R for a power series.

(b)[2] Give an example of a power series which converges *only* on the closed interval $[-1, 1]$ (note the endpoints are included).

(b) [5] Given a function $f(x)$ suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

on the interval $(a-R, a+R)$. Explain why the form of the coefficients a_n are uniquely determined by the derivatives of f at $x = a$.

(c) [3] (i) Find the second order Taylor polynomial $T_2(x)$ for $f(x) = \cos x$ near $x = 0$.

(ii) [5] Use the Remainder Theorem to estimate the maximum error in the approximation of $\cos x$ for $|x| < .2$ by the polynomial $T_2(x)$ in (i) above.

(2)[15] (a) [4] A series $\sum_{n=0}^{\infty} a_n$ converges absolutely if

$$\sum_{n=0}^{\infty} |a_n|$$

converges. Give an example of a series that converges but does not converge absolutely.

(b) [3] Determine whether or not

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by using the integral test.

(c) [8] Explain the truth or falsity of the following two statements, giving reasons and examples (or counterexamples):

(i) If $0 < a_n \leq b_n$ and $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges.

(iii) If $\sum_{n=0}^{\infty} a_n^2$ converges then $\sum_{n=0}^{\infty} a_n$ converges.

(3) [20 points] Determine whether the following series converge. State precisely your reasons.

(a)

$$\sum_{n=1}^{\infty} \frac{n^3 + 2n + 1}{n^4 + n + 2}$$

(b)

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$$

(d)

$$\sum_{n=0}^{\infty} \frac{2^{1/n}}{(1.2)^n}$$

(4) [15 points] (i) Find the 5th order Taylor polynomial about $a = 0$ of the function

$$x^2 e^{2x}$$

Hint: find the Taylor polynomial expansion of e^{2x} *first.*

(ii) Using (i) or otherwise find $f^{(5)}(0)$ of $f(x) = x^2 e^{2x}$.

(5) [10 points] (i) Find the power series expansion of the function

$$\frac{1}{1+x^2}$$

about $a = 0$.

(ii) Using (i) or otherwise find the Taylor series expansion of

$$\arctan(x)$$

about $a = 0$ stating carefully any theorems you may use about integrating or differentiating power series within their radius of convergence.