Math 1450, Honor Calculus Practice5, Fall 2016.

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1. A number a is called a fixed point of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Contradict proof: Assume f has ~~two~~ fixed points a, b, $a \neq b$.

Without loss of generality, we could let $a < b$, then by MVT, there is a number c such that $c \in (a, b)$ and

$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{b-a}{b-a} = 1.$$

However, $f'(x) \neq 1$ for all x, so the assumption we made is wrong and f has at most one fixed point.

2. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leq |a - b| \quad \text{for all } a \text{ and } b.$$

Case 1. $a > b$ (same as $a < b$)

let $f(x) = \sin(x)$, we obtain,

By MVT, there is $c \in (b, a)$ such that

$$\frac{\sin(a) - \sin(b)}{a-b} = \frac{f(a) - f(b)}{a-b} = f'(c) = \cos(c)$$

Take "| |" on both sides, we get

$$\left|\frac{\sin(a) - \sin(b)}{a-b}\right| = |\cos(c)| \leq 1$$

$$\Rightarrow |\sin(a) - \sin(b)| \leq |a-b|$$

Case 2, if $a = b$.

then

$\sin(a) - \sin(b) = 0 = a - b$.

3. Use mathematical induction to prove for all $n \geq 1$

To prove $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. is true $\forall n \geq 1$.

First, as $n=1$, we have

$\boxed{LHS = 1 \quad , \quad RHS = \frac{1\cdot(1+1)(2+1)}{6} = \frac{6}{6} = 1 \Rightarrow LHS = RHS}$

Then, assume as $n=k$, we have $\boxed{1^2+2^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}}$

$2k^2+7k+6$

So, as $n=k+1$, we have

$LHS = 1^2+2^2+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = (k+1)\left[\frac{k(2k+1)}{6} + k+1\right]$ $\frac{1}{2} \times \frac{2}{3}$

$= (k+1)\frac{(k+2)(2k+3)}{6}$

and $RHS = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$\Rightarrow LHS = RHS$.

So, by math induction, this statement is true $\forall n \geq 1$.

4. Use mathematical induction to prove for all $n \geq 1$

To prove $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$. is true $\forall n \geq 1$.

First, as $n=1$, we have

$\boxed{LHS = 1 \times 2 = 2, \ RHS = \frac{1\cdot(1+1)(1+2)}{3} = \frac{6}{3} = 2 \Rightarrow LHS = RHS}$

So when $n=1$, this statement is true.

Then assume as $n=k$, we have $\boxed{1\times2+2\times3+\cdots+k\times(k+1) = \frac{k(k+1)(k+2)}{3}}$

So, as $n=k+1$, we have

$LHS = 1\times2+2\times3+\cdots+k\times(k+1)+(k+1)\times(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$= (k+1)(k+2)\left[\frac{k}{3}+1\right] = \frac{(k+1)(k+2)(k+3)}{3}$.

$RHS = \frac{(k+1)(k+1+1)(k+1+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3} \Rightarrow LHS = RHS$

when $n=k+1$, this statement is right.

Thus, by math induction, this statement is right $\forall n \geq 1$.