Intermediate Value Theorem (see textbook p.126)

Suppose that f is continuous on $[a,b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there is a number c in (a,b) such that $f(c) = N$.

Math 1450, Honor Calculus Practice4, Fall 2016.

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1. Show that the equation $4x^5 + x^3 + 2x + 1 = 0$ has exactly one real root.

Contradict Proof let $f(x) = 4x^5 + x^3 + 2x + 1$

Assume f has two root a, b, $a \neq b$, $a < b$.

So $f(a) = f(b) = 0$.

However, $f'(x) = 20x^4 + 3x^2 + 2 > 0$ means f is a strickly increasing function.

and there are no such a, b with $f(a) = f(b)$.

So the assumption is wrong and f has at most one root.

By Intermediate Value Thm.

we have $f(0) = 1 > 0$ and $f(-1) = -6 < 0$

So there is $c \in (-1, 0)$ such that $f(c) = 0$

which means f has exactly one root.

2. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

Since $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ and $f''(x) = \lim_{h \to 0} \dfrac{f'(x) - f'(x-h)}{h}$

$$f'(x-h) = \lim_{h \to 0} \frac{f(x-h+h) - f(x-h)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

$$\text{So } f''(x) = \lim_{h \to 0} \frac{\dfrac{f(x+h) - f(x)}{h} - \dfrac{f(x) - f(x-h)}{h}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

3. Investigate the family of polynomials given by the equation $f(x) = 2x^3 + cx^2 + 2x$. For what values of c does the curve have maximum and minimum points?

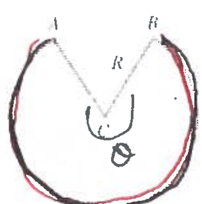
Finding critical point of $f(x)$. we have $f'(x)=6x^2+2cx+2=0$

$6x^2+2cX+2=0$ is solvable $\Leftrightarrow \sqrt{4c^2-48}>0$

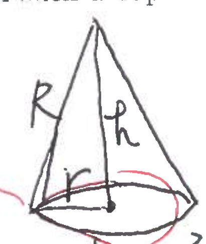
$\Rightarrow 4c^2-48>0 \Rightarrow c^2-12>0 \Rightarrow (c-2\sqrt{3})(c+2\sqrt{3})>0$

$+ \quad - \quad +$; $-2\sqrt{3}$, $2\sqrt{3}$ $\Rightarrow c>2\sqrt{3}$ or $c<-2\sqrt{3}$.

4. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.

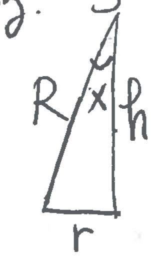


$= 2R\pi\frac{\theta}{2\pi} = 2\pi r$



$\Rightarrow r(\theta)=\frac{R\theta}{2\pi}$ and $h(\theta)=\sqrt{R^2-r^2(\theta)}$

But it's too complicated! so let us use another approaching. $V=\frac{1}{3}\pi r^2 h$



let $h(x)=R\cos(x)$, $r(x)=R\sin(x)$

$\frac{dh}{dx}=-R\sin(x)$, $\frac{dr}{dx}=R\cos(x)$

and $V=\frac{1}{3}\pi r^2 h$, $\frac{dV}{dx}=\frac{1}{3}\pi 2r\cdot\frac{dr}{dx}h + \frac{1}{3}\pi r^2\frac{dh}{dx}$

we have $\frac{dV}{dx}=\frac{1}{3}\pi\cdot 2R\sin(x)\cdot R\cos(x)\cdot R\cos(x)+\frac{1}{3}\pi R^2\sin^2(x)\cdot(-R\sin(x))=0$

$x\in[0,2\pi]$

$\Rightarrow \sin(x)\cos^2(x)-\sin^3(x)=0 \Rightarrow \sin(x)\big(\cos(x)+\sin(x)\big)\big(\cos(x)-\sin(x)\big)=0$

$\Rightarrow \sin(x)=0 \Rightarrow x=0, 2\pi \Rightarrow V=0$

$\sin(x)=-\cos(x) \Rightarrow x=\frac{3\pi}{4}, \frac{5\pi}{4} \Rightarrow V<0$ ✗

$\sin(x)=\cos(x)$ $\quad x=\frac{\pi}{4}, \frac{7\pi}{4} \Rightarrow V=\frac{1}{3}\pi\cdot\left(\frac{\sqrt{2}}{2}\right)^2\cdot\frac{\sqrt{2}}{2}$ abs max

$=\frac{\pi}{12}\sqrt{2}$.