Math 1450, Honor Calculus Practice15, Fall 2016.

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Use the properties of integrals to verify the following inequalities.

1. $\int_0^{\frac{\pi}{2}} x\sin(x)\,dx \le \frac{\pi^2}{8}$.

since $\sin(x) \le 1 \Rightarrow x\cdot\sin(x) \le x$

Then $\int_0^{\frac{\pi}{2}} x\sin(x)dx \le \int_0^{\frac{\pi}{2}} x\,dx = \frac{x^2}{2}\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}$

2. $\frac{\sqrt{2}\pi}{24} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos(x)\,dx \le \frac{\sqrt{3}\pi}{24}$.

since on $[\frac{\pi}{6}, \frac{\pi}{4}]$ $\frac{\sqrt{2}}{2} \le \cos(x) \le \frac{\sqrt{3}}{2}$, we have

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{2}}{2}dx \le \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos(x)\,dx \le \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{3}}{2}\,dx$

$\Rightarrow \underbrace{\frac{\sqrt{2}}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}_{=\frac{\sqrt{2}\pi}{24}} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos(x)dx \le \frac{\sqrt{3}}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{24}\pi$.

3. $\int_1^3 \sqrt{x^4+1}\,dx \ge \frac{26}{3}$.

since $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$ on $[1,3]$

Then $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2dx = \frac{x^3}{3}\Big|_1^3 = \frac{26}{3}$

(let $f(x)=\sqrt{1+x^2}$, $f'(x)=\frac{2x}{2}\frac{1}{\sqrt{1+x^2}}=0$. $\underset{\text{critical pt.}}{x=0}$ $f'(x)$: $-$ on $(-1,0)$, $+$ on $(0,1)$; min @ $x=0$,

$f(0)=\sqrt{1}=1 \to$ abs. min.

$f(-1)=\sqrt{2}$, $f(1)=\sqrt{2}$) abs. max.

4. $2 \le \int_{-1}^{1} \sqrt{1+x^2}\,dx \le 2\sqrt{2}$.

Since $1 \le \sqrt{1+x^2} \le \sqrt{2}$ on $[-1,1]$

$$\Rightarrow \underbrace{\int_{-1}^{1} dx}_{=2} \le \int_{-1}^{1}\sqrt{1+x^2}\,dx \le \underbrace{\int_{-1}^{1}\sqrt{2}\,dx}_{=2\sqrt{2}}$$

5. $\int_{10}^{15} \frac{t^3}{t^6+t^2-1}\,dt \le \frac{3}{1000}$.

since $t^2-1>0$ as $t\in[10,15]$, we have

$$\frac{t^3}{t^6+t^2-1} \le \frac{t^3}{t^6} = \frac{1}{t^3} \Rightarrow \int_{10}^{15}\frac{t^3}{t^6+t^2-1}\,dt \le \int_{10}^{15}\frac{dt}{t^3}$$

and $\int_{10}^{15}\frac{dt}{t^3} = -\frac{1}{2}\frac{1}{t^2}\Big|_{10}^{15} = -\frac{1}{2}\left(\frac{1}{225}-\frac{1}{100}\right) = -\frac{1}{2}\left(\frac{-5}{900}\right) = \frac{1}{360}$

so $\int_{10}^{15}\frac{t^3}{t^6+t^2-1}\,dt \le \frac{1}{360} < \frac{3}{1000}$.

6. $\frac{3}{4} \le \int_0^1 \frac{1}{1+t^4}\,dt \le \frac{9}{10}$.

Since $\frac{1}{1+t^2} \le \frac{1}{1+t^4}$ on $[0,1]$

so $\int_0^1 \frac{dt}{1+t^2} \le \int_0^1 \frac{dt}{1+t^4}$

$\int_0^1 \frac{dt}{1+t^2} = \arctan(t)\Big|_0^1 = \frac{\pi}{4}-0 = \frac{\pi}{4} \ge \frac{3}{4}$

$\Rightarrow \frac{3}{4} \le \int_0^1 \frac{dt}{1+t^4}$

we have $1-\int_0^1\frac{dt}{1+t^4}$

$= \int_0^1 dt - \int_0^1 \frac{dt}{1+t^4} = \int_0^1 \frac{t^4}{1+t^4}$

$\frac{1}{1+t^4} \ge \frac{1}{2}$

$\ge \int_0^1 \frac{t^4}{1+1}\,dt = \frac{1}{2}\int_0^1 t^4\,dt$

$= \frac{1}{2}\frac{t^5}{5}\Big|_0^1 = \frac{1}{10}$

$\Longrightarrow 1-\int_0^1\frac{dt}{1+t^4} \ge \frac{1}{10}$

$\Rightarrow \int_0^1 \frac{dt}{1+t^4} \le \frac{9}{10}$.