Honors Calculus, Math1450. Assignment 8, Solutions.

(1)

(a) $\sum_{n\geq 2} \frac{1}{(\sqrt{3})^n} = \frac{\frac{1}{(\sqrt{3})^2}}{1-\frac{1}{\sqrt{3}}} = \frac{1}{3} \cdot \frac{\sqrt{3}}{\sqrt{3}-1} = \frac{\sqrt{3}(\sqrt{3}+1)}{3\cdot(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{3+\sqrt{3}}{6}$

$\left(\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r} \text{ for } |r|<1\right)$.

(b) Given $\sum_{n=1}^{\infty} |x-1|^n$, Using root test, we have

if $\sqrt[n]{|x-1|^n} < 1$ then $\sum_{n=1}^{\infty} |x-1|^n$ converges

so $\sqrt[n]{|x-1|^n} = |x-1| < 1 \Rightarrow 0 < x < 2$

as $x=0$ $\sum_{n=1}^{\infty} |-1|^n = \sum_{n=1}^{\infty} 1$ div.

& as $x=2$ $\sum_{n=1}^{\infty} |1|^n = \sum_{n=1}^{\infty} 1$ div.

(2) By geometric series, we have $\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r}$ for $|r|<1$.

Given $|x|<1$ we have $|x^2|<1$. let $r=-x^2$, $a_0=1$, then

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} 1\cdot(-x^2)^n = 1+(-x^2)+(-x^2)^2+(-x^2)^3+\cdots+(-x^2)^n+\cdots$$
$$= 1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots.$$

For $\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{4}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{4} (-1)^n \left(\frac{1}{4}\right)^{2n} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{4}\right)^{2n}$

So, now $x=\frac{1}{4}$, and $\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{4}\right)^{2n+1} = \frac{1}{4} \cdot \frac{1}{1+\left(\frac{1}{4}\right)^2} = \frac{1}{4} \cdot \frac{16}{17} = \frac{4}{17}$.

(3)

(a) Given $\sum_{n\geq 1} \frac{\ln(n)}{n}$. By integral test, since $\int_1^{\infty} \frac{\ln(x)}{x}dx = \lim_{a\to\infty} \left(\frac{u^2}{2}\right)\Big|_1^a$ diverges.

Thus $\sum_{n\geq 1} \frac{\ln(n)}{n}$ diverges.

(b) Given $\sum_{n\geq 1} \frac{3n(n-1)}{n^2+1}$, By divergent test,

since $\lim_{n\to\infty} \frac{3n(n-1)}{n^2+1} = 3 \neq 0$, thus $\sum_{n\geq 1} \frac{3n(n-1)}{n^2+1}$ diverges

(c) Given $\sum_{j=1}^{\infty} \frac{\cos(\pi j)}{j^2}$, since $\cos(\pi j) = (-1)^j$ for $j \in \mathbb{N}$.

Then $\sum_{j=1}^{\infty} \frac{\cos(\pi j)}{j^2} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2}$. By alternating series test,

since $\frac{1}{j^2} \to 0$ as $j \to \infty$, Then $\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2}$ converges.

and $\frac{1}{j^2} > \frac{1}{(j+1)^2}$ $(b_n > b_{n+1})$

(or check abs. convergent by steve)

(d) Given $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+1}$. let $a_n = \frac{\sqrt{n}}{n^2+1}$ and $b_n = \frac{\sqrt{n}}{n^2}$

since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{\sqrt{n}}{n^2}} = 1 > 0$. and $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

converges by p-series test, so by limit Comparison test,

$\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges.

(4).

(a) §11.2 <u>58</u>

(a) distance $= H + 2rH + 2r^2H + 2r^3H + \cdots$

$= H + 2H(r + r^2 + r^3 + \cdots)$

$= H + 2H \cdot \dfrac{r}{1-r}$ (by geometric series and $0<r<1$)

$= H\left[1 + \dfrac{2r}{1-r}\right] = H \cdot \dfrac{1+r}{1-r}$

(b) since the falling distance $h = \frac{1}{2}gt^2$, then $t = \sqrt{\dfrac{2h}{g}}$.

Then, by (a), the total time is $\sqrt{\dfrac{2H}{g}} + 2\sqrt{\dfrac{2H}{g}r} + 2\sqrt{\dfrac{2H}{g}r^2} + \cdots$

$= \sqrt{\dfrac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r^2} + 2\sqrt{r^3} + \cdots\right] = \sqrt{\dfrac{2H}{g}}\left[1 + 2\cdot\dfrac{\sqrt{r}}{1-\sqrt{r}}\right] = \sqrt{\dfrac{2H}{g}} \cdot \dfrac{1+\sqrt{r}}{1-\sqrt{r}}$.

(c) Since $\dfrac{d^2x}{dt^2} = -g \Rightarrow \dfrac{dx}{dt} = -gt + v_0 \Rightarrow x = -g\dfrac{t^2}{2} + v_0 t + x_0$

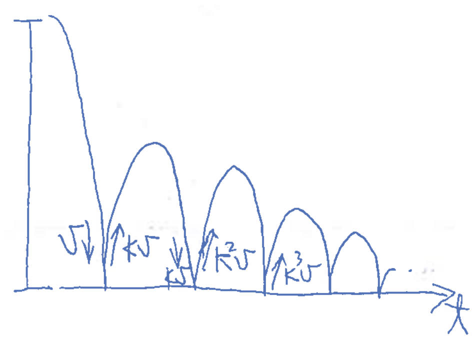
and $v = +gt$ where $t = \sqrt{\dfrac{2H}{g}}$,

$rH = kvt - g\dfrac{1}{2}t^2$ where $t = \sqrt{\dfrac{2rH}{g}}$

$\Rightarrow rH = k \cdot g\sqrt{\dfrac{2H}{g}} \cdot \sqrt{\dfrac{2rH}{g}} - g\dfrac{1}{2}\cdot\dfrac{2rH}{g}$

$= k\,2H\sqrt{r} - rH$.

$\Rightarrow 2rH = 2Hk\sqrt{r} \Rightarrow \sqrt{r} = k$.



By (b), the ~~total~~ time is $\sqrt{\dfrac{2H}{g}}\,\dfrac{1+k}{1-k}$.

(4)

(b) Given $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. By integral test, since

$\int_1^{\infty} \frac{dx}{x\ln(x)} = \lim_{b\to\infty} \lim_{a\to 1} \left(\ln(\ln x)\right)_a^b$ diverges, then $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

(c) Given $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$. By integral test, since

$\int_1^{\infty} \frac{dx}{x\cdot \ln(x)^2} = \int_1^{\infty} \frac{dx}{x\cdot 2\ln(x)} = \frac{1}{2}\int_1^{\infty} \frac{dx}{x\cdot\ln(x)}$ diverges, then $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$ diverges

(5)

(a) Given $\sum_{n=3}^{\infty} \frac{1}{n^2-7}$. let $a_n = \frac{1}{n^2-7}$, $b_n = \frac{1}{n^2}$. since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2-7} = 1 > 0$

and $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges by p-series, then by limit comparison test,

$\sum_{n=3}^{\infty} \frac{1}{n^2-7}$ converges

(b) Given $\sum_{n=1}^{\infty} \frac{n}{n^4+2}$. let $a_n = \frac{n}{n^4+2}$, $b_n = \frac{1}{n^3}$. (or $0 < \frac{n}{n^4+2} \le \frac{1}{n^3}$ then use direct comparison)

since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^4+2}}{\frac{1}{n^3}} = 1 > 0$, and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

by p-series test, then by limit comparison test.

$\sum_{n=1}^{\infty} \frac{n}{n^4+2}$ converges.

(5).

(c) Given $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$. let $a_n = \frac{n^2}{e^n}$, since, by ratio test,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{1}{e}\left(\frac{n+1}{n}\right)^2 = \frac{1}{e} < 1.$$

So $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ converges.

(d) Given $\sum_{n=1}^{\infty} \frac{2^n+3}{3^n+n}$. By comparison test, since

$$0 < \frac{2^n+3}{3^n+n} < \frac{2^n+3}{3^n} < \frac{2^n+2^n}{3^n} = 2\cdot\left(\frac{2}{3}\right)^n$$ and $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^n$ converges $\left(\frac{2}{3}<1\right)$.

Thus $\sum_{n=1}^{\infty} \frac{2^n+3}{3^n+n}$ converges

(e) Given $\sum_{n=1}^{\infty} \frac{n}{(n^3+n)^{\frac{1}{2}}}$. let $a_n = \frac{n}{(n^3+n)^{\frac{1}{2}}}$. By limit comparison test,

let $b_n = \frac{n}{(n^3)^{\frac{1}{2}}}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{(n^3+n)^{\frac{1}{2}}} \cdot \frac{(n^3)^{\frac{1}{2}}}{n} = 1 > 0$

and $\sum_{n=1}^{\infty} \frac{n}{(n^3)^{\frac{1}{2}}}$ diverges (by p-series), then

$\sum_{n=1}^{\infty} \frac{n}{(n^3+n)^{\frac{1}{2}}}$ diverges.

(f) Given $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$. let $a_n = \frac{e^{\frac{1}{n}}}{n}$, $b_n = \frac{1}{n}$. By limit comparison test,

since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{e^{\frac{1}{n}}}{n}}{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}} = 1 > 0$ and

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Thus $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$ diverges.