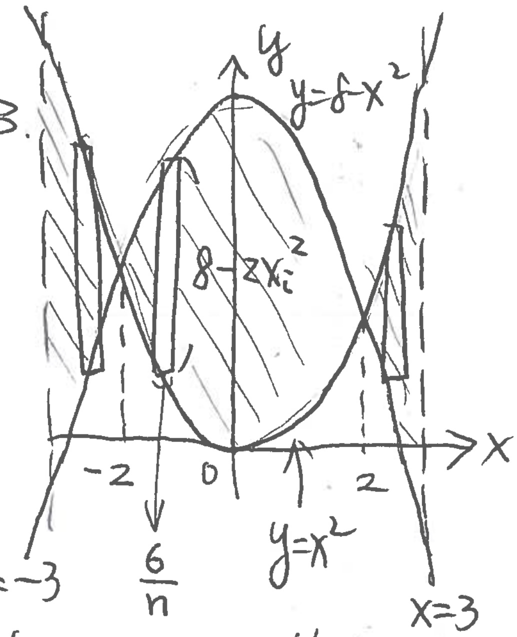
Honors Calculus, Math 1450, Assignment 6 Solution

§6.1

18. Given $y=8-x^2$, $y=x^2$, $x=-3$, $x=3$.



$$\text{Area}=\int_{-3}^{-2} x^2-(8-x^2)\,dx+\int_{-2}^{2}(8-2x^2)\,dx+\int_{2}^{3}x^2-(8-x^2)\,dx$$

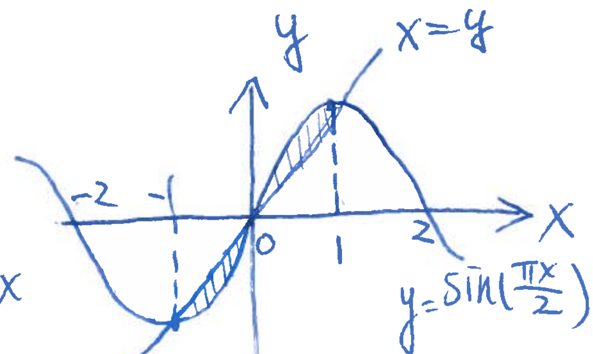
symmetric region

$$=2\left[\int_0^2 (8-2x^2)\,dx+\int_2^3 (2x^2-8)\,dx\right]$$

$$=2\left[8x-\frac{2}{3}x^3\Big|_0^2+\frac{2}{3}x^3-8x\Big|_2^3\right]=2\left[16-\frac{16}{3}+18-24-\frac{16}{3}+16\right]$$

$$=2\left[\frac{64}{3}-6\right]=\frac{92}{3}$$

22. Given $y=\sin\left(\frac{\pi x}{2}\right)$, $y=x$.



$$\text{Area}=\int_{-1}^{0}\left(x-\sin\left(\frac{\pi x}{2}\right)\right)dx+\int_0^1\left|\sin\left(\frac{\pi x}{2}\right)-x\right|dx$$

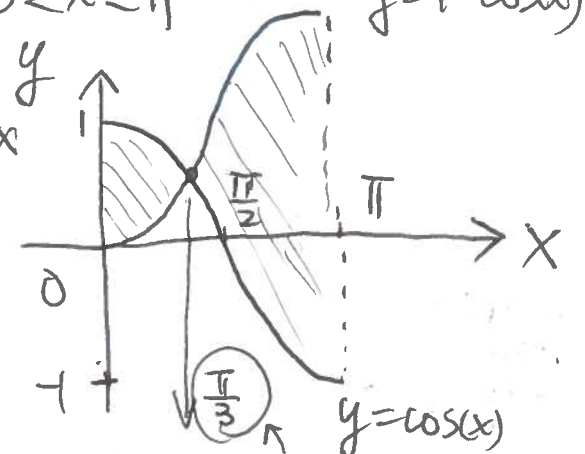
symmetric region

$$=2\left[\int_0^1\left(\sin\left(\frac{\pi}{2}x\right)-x\right)dx\right]=2\cdot\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)-\frac{x^2}{2}\right]_0^1$$

$$=2\cdot\left[-\frac{2}{\pi}(0-1)-\left(\frac{1}{2}-0\right)\right]=\frac{4}{\pi}-1$$

§6.1

24. Given $y=\cos(x)$, $y=1-\cos(x)$, $0\le x\le\pi$



$$\text{Area}=\int_0^{\frac{\pi}{3}}\cos(x)-(1-\cos(x))\,dx+\int_{\frac{\pi}{3}}^{\pi}(1-\cos(x))-\cos(x)\,dx$$

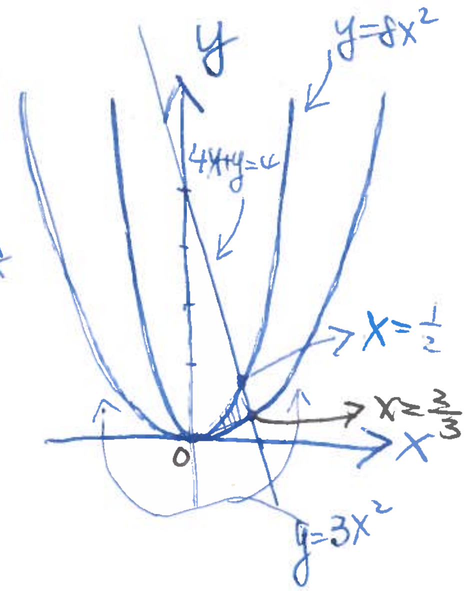
$$=\int_0^{\frac{\pi}{3}}(2\cos(x)-1)\,dx+\int_{\frac{\pi}{3}}^{\pi}(1-2\cos(x))\,dx$$

$$=\Big[2\sin(x)-x\Big]_0^{\frac{\pi}{3}}+\Big[x-2\sin(x)\Big]_{\frac{\pi}{3}}^{\pi}$$

$$=2\cdot\frac{\sqrt{3}}{2}-\frac{\pi}{3}+(\pi-2\cdot 0)-\left(\frac{\pi}{3}-\sqrt{3}\right)=2\sqrt{3}+\frac{\pi}{3}.$$

$$\boxed{\begin{aligned}&1-\cos(x)=\cos(x)\\&\cos(x)=\tfrac{1}{2}\Rightarrow x=\tfrac{\pi}{3}\end{aligned}}$$

28. Given $y=3x^2$, $y=8x^2$, $4x+y=4$, $x\ge 0$



Intersection of $y=8x^2$ and $4x+y=4$:

$$4x+8x^2-4=0\Rightarrow 2x^2+x-1=0\Rightarrow x=\frac{1}{2},\ \cancel{-1}$$

Intersection of $y=3x^2$ and $4x+y=4$,

$$4x+3x^2-4=0\Rightarrow x=\cancel{-2}\text{ or }\frac{2}{3}$$

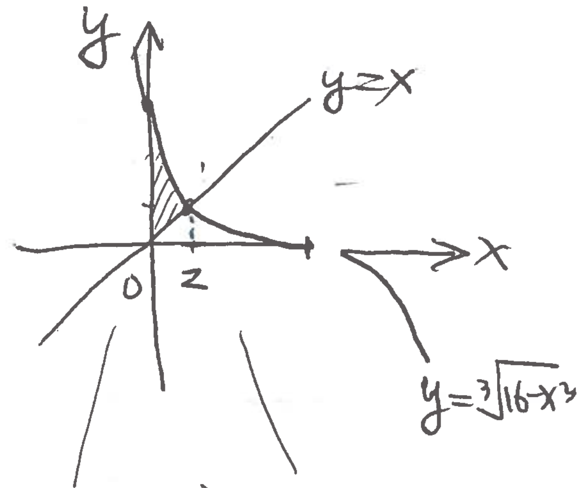
$$\text{Area}=\int_0^{\frac{1}{2}}(8x^2-3x^2)\,dx+\int_{\frac{1}{2}}^{\frac{2}{3}}4-4x-(3x^2)\,dx$$

$$=\left[\frac{5}{3}x^3\right]_0^{\frac{1}{2}}+\left[4x-2x^2-x^3\right]_{\frac{1}{2}}^{\frac{2}{3}}=\frac{5}{24}+\left[4\left(\frac{2}{3}-\frac{1}{2}\right)-2\left(\frac{4}{9}-\frac{1}{4}\right)-\left(\frac{8}{27}-\frac{1}{8}\right)\right]$$

$$=\frac{5}{24}+\left[\frac{2}{3}-\frac{7}{18}-\frac{37}{216}\right]=\frac{17}{54}$$

§6.1

34. Given $y=\sqrt[3]{16-x^3}$, $y=x$, $x=0$.



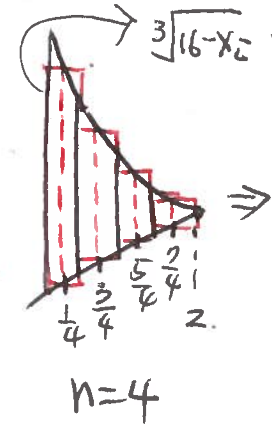
Riemann Sum as $n=4$:

let $f(x_i)=\sqrt[3]{16-x_i^3}-x_i$

Then $R=f(\frac{1}{4})\cdot\overset{\text{width of the bars}}{\frac{2}{4}}+f(\frac{3}{4})\cdot\frac{2}{4}+f(\frac{5}{4})\cdot\frac{2}{4}$

$+f(\frac{7}{4})\cdot\frac{2}{4}$

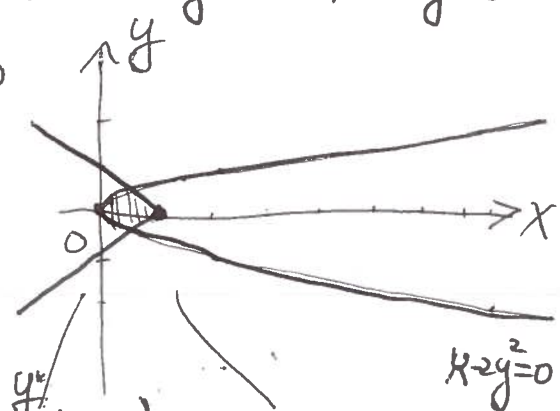
$=\frac{1}{2}\left[\sqrt[3]{16-(\frac{1}{4})^3}+\sqrt[3]{16-(\frac{3}{4})^3}+\sqrt[3]{16-(\frac{5}{4})^3}+\sqrt[3]{16-(\frac{7}{4})^3}-\frac{1}{4}-\frac{3}{4}-\frac{5}{4}-\frac{7}{4}\right]\cong 2.8144$



40. Given $x-2y^2\geq 0$, $1-x-|y|\geq 0 \Rightarrow \begin{cases}1-x-y\geq 0 & \text{for } y>0\\ 1-x+y\geq 0 & \text{for } y<0\end{cases}$

Intersection point(s) of $x-2y^2=0$ and $1-x-y=0$

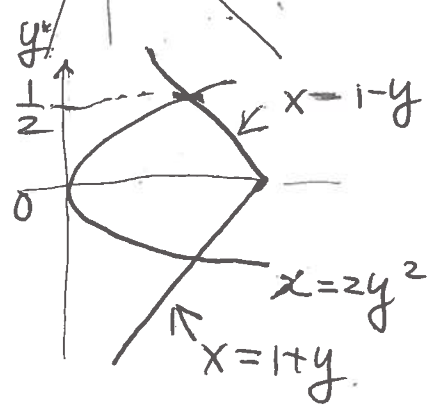
as $y>0$.: $2y^2+y-1=0 \Rightarrow y=\frac{1}{2}$



$\underset{\substack{\uparrow\\ \text{symmetric region}}}{\text{Area}=}\ 2\cdot\int_0^{\frac{1}{2}}(1-y)-2y^2\,dy$

$=2\left[y-\frac{y^2}{2}-\frac{2}{3}y^3\right]_0^{\frac{1}{2}}$

$=2\left[\frac{1}{2}-\frac{1}{8}-\frac{1}{12}\right]=\boxed{\frac{7}{12}}$



§6.1

48. Given $y=x^2$, its tangent line at $(1,1)$ and x-axis.

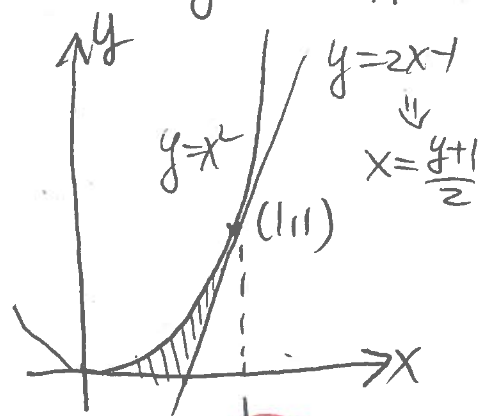
To Find the tangent line of $y=x^2$ @ $(1,1)$, we have

$f(x)=y=x^2$, $f'(x)=2x \Rightarrow$ slope @ $(1,1)$ is 2. $\Rightarrow$ tangent line $y-1=2(x-1)$.

$\Rightarrow y=2x-1.$

$$\text{Area}=\int_0^1\left(\frac{y+1}{2}-\sqrt{y}\right)dy=\frac{y^2}{4}+\frac{y}{2}-\frac{2}{3}y^{\frac{3}{2}}\Big|_0^1$$

$$=\frac{1}{4}+\frac{1}{2}-\frac{2}{3}=\boxed{\frac{1}{12}}$$



$$\text{or}=\int_0^{\frac{1}{2}}x^2\,dx+\int_{\frac{1}{2}}^1 x^2-2x+1\,dx$$

$$=\frac{x^3}{3}\Big|_0^{\frac{1}{2}}+\left(\frac{x^3}{3}-x^2+x\right]_{\frac{1}{2}}^1=\frac{1}{24}+\left[\frac{1}{3}\left(\frac{7}{8}\right)-\frac{3}{4}+\frac{1}{2}\right]=\boxed{\frac{1}{12}}$$

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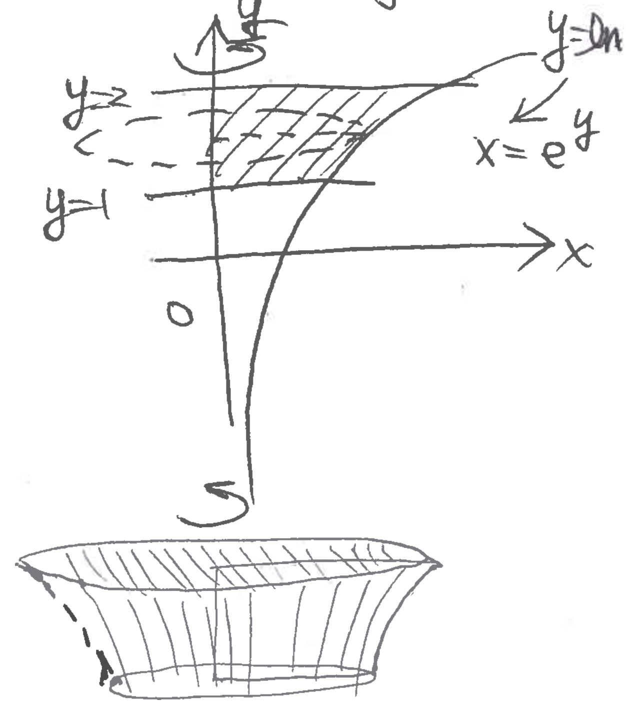
§6.2. Given

6. $y=\ln x$, $y=1$, $y=2$, $x=0$ and find the rotating volume about y-axis.

$$V_r=\int_1^2\pi\cdot\left(e^y\right)^2dy=\pi\int_1^2 e^{2y}dy$$

$$=\pi\,\frac{e^{2y}}{2}\Big|_1^2=\frac{\pi}{2}\left(e^4-e^2\right)$$



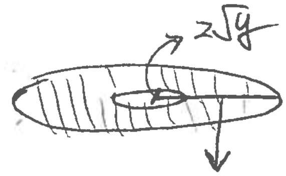
§6.2

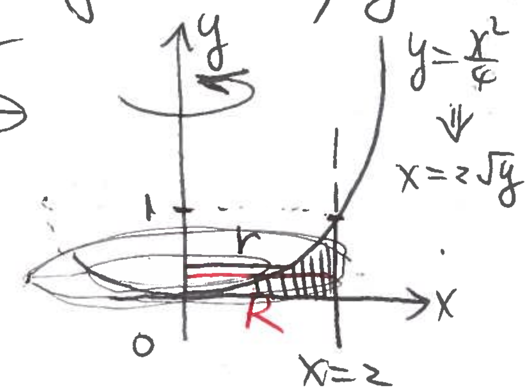
10. Given $y=\frac{x^2}{4}$, $x=2$, $y=0$, and find the rotating volume by y-axis.

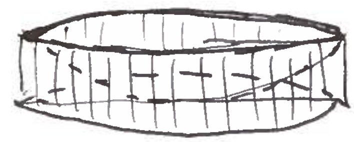
$$V_r=\int_0^1 \pi\left(2^2-(2\sqrt{y})^2\right)dy$$

$$=\pi\int_0^1 (4-4y)\,dy=\pi\left[4y-2y^2\right]_0^1$$

$$=\pi[4-2]=\underline{2\pi}.$$







14. Given $y=\frac{1}{x}$, $y=0$, $x=1$, $x=3$ and find the rotating volume about $y=-1$

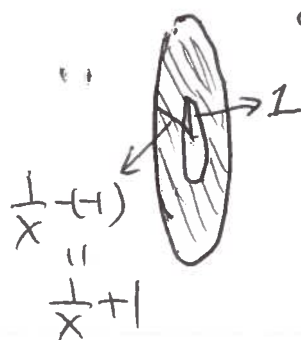
$$V_r=\int_1^3 \pi\left[\left(\tfrac{1}{x}+1\right)^2-1^2\right]dx$$

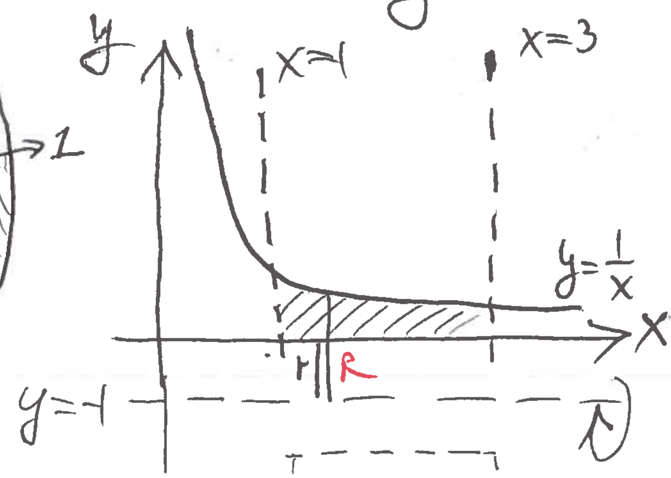
$$=\pi\int_1^3\left[\frac{1}{x^2}+\frac{2}{x}\right]dx$$

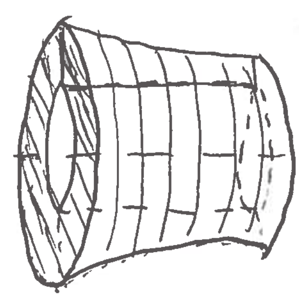
$$=\pi\left[-\frac{1}{x}+2\ln|x|\right]_1^3$$

$$=\pi\left[-\frac{1}{3}+1+2\ln 3-\cancelto{0}{2\ln 1}\right]=\underline{\pi\left(2\ln 3+\frac{2}{3}\right)}$$

$\underline{\underline{\ln 1=0}}$

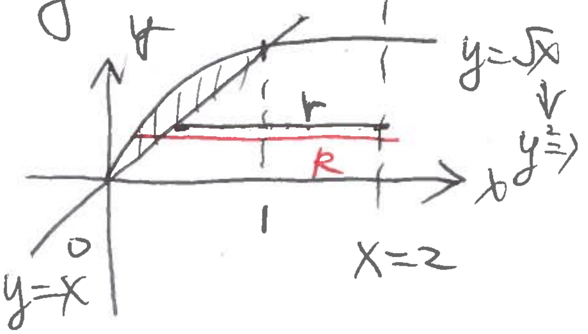






§6.2

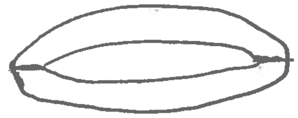
16. Given $y=x$, $y=\sqrt{x}$ and find the rotating volume about $x=2$



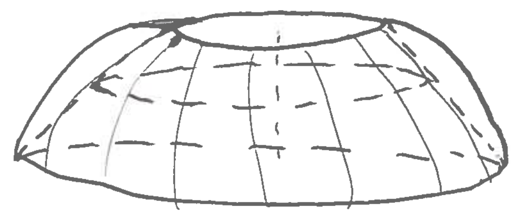
the radius of inner circle: $2-y$

the radius of outer circle:

$2-y^2$



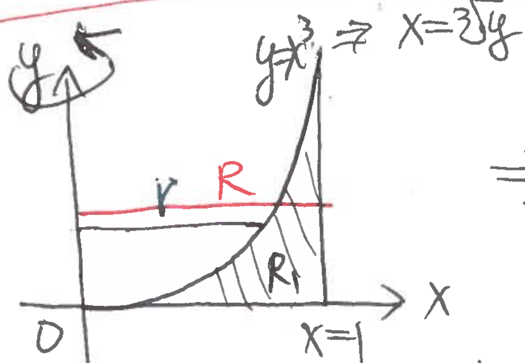
$$V_R=\int_0^1 \pi\left[(2-y^2)^2-(2-y)^2\right]dy$$



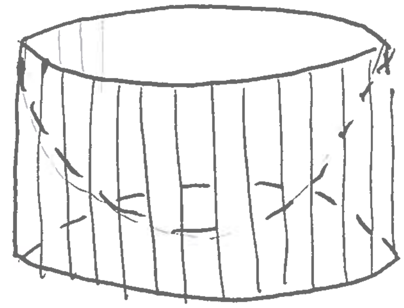
$$=\pi\int_0^1 \{4-4y^2+y^4-(4-4y+y^2)\}dy$$

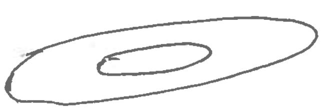
$$=\pi\int_0^1 (4y-5y^2+y^4)\,dy=\pi\left[2y^2-\frac{5}{3}y^3+\frac{y^5}{5}\right]_0^1=\pi\left[2-\frac{5}{3}+\frac{1}{5}\right]=\frac{8}{15}\pi$$

20.



$\Longrightarrow$





the radius of inner circle: $\sqrt[3]{y}$

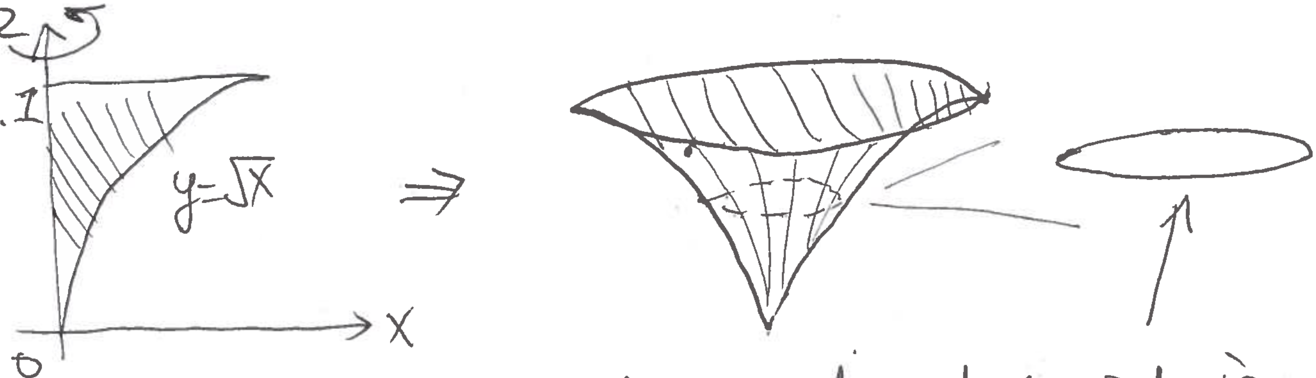
the radius of outer circle 1

$$V_R=\pi\int_0^1 1-(\sqrt[3]{y})^2\,dy=\pi\left[y-\frac{3}{5}y^{\frac{5}{3}}\right]_0^1=\pi\left[1-\frac{3}{5}\right]=\frac{2}{5}\pi$$

$(\sqrt[3]{y})^2 \to y^{\frac{2}{3}}$

§6.2

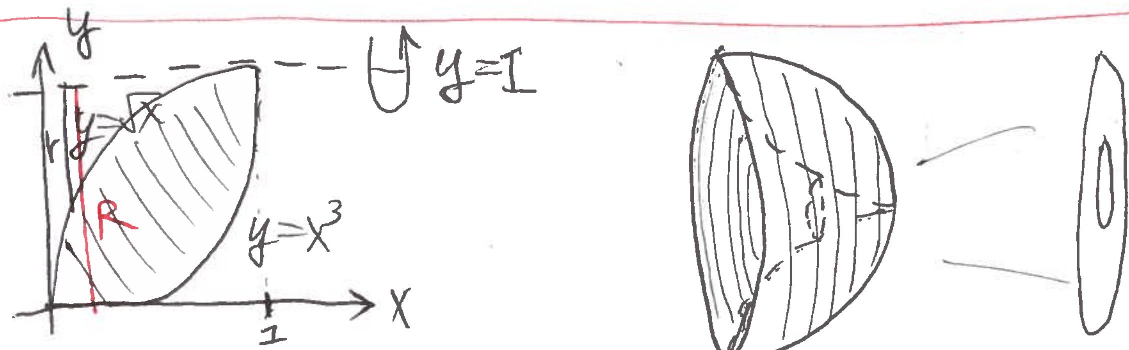
24.



there is no inner circle and the radius of the circle is y^2

$$V_R = \pi \int_0^1 (y^2)^2 dy = \pi \frac{y^5}{5}\Big|_0^1 = \underline{\frac{\pi}{5}}.$$

30.



the radius of inner circle is $1-\sqrt{x}$, and the radius of outer circle is $1-x^3$. then

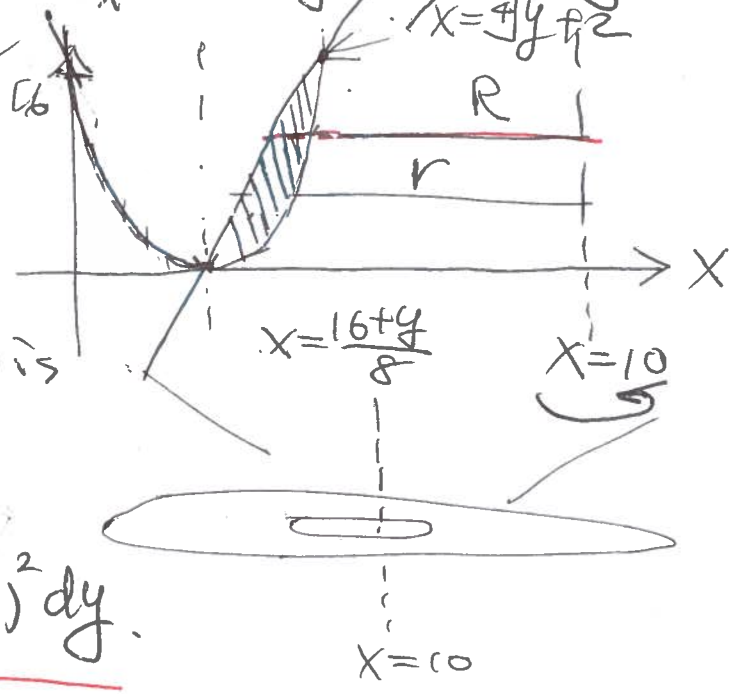
$$V_R = \pi \int_0^1 (1-x^3)^2 - (1-\sqrt{x})^2 dx = \pi \int_0^1 (1-2x^3+x^6) - (1-2\sqrt{x}+x)\, dx$$

$$= \pi \int_0^1 (2\sqrt{x} - 2x^3 - x + x^6) dx = \pi \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^4}{2} - \frac{x^2}{2} + \frac{x^7}{7} \right]_0^1$$

$$= \pi \left[\frac{4}{3} - \frac{1}{2} - \frac{1}{2} + \frac{1}{7} \right] = \underline{\frac{10}{21}\pi}$$

§6.2

32. Given $y=(x-2)^4$, $8x-y=16$ and find the integral of rotating volume about $x=10$.



the radius of outer circle R is

$10-\frac{16+y}{8}=8-\frac{y}{8}$

and the radius of inner circle r is

$10-\sqrt[4]{y}-2=8-\sqrt[4]{y}$

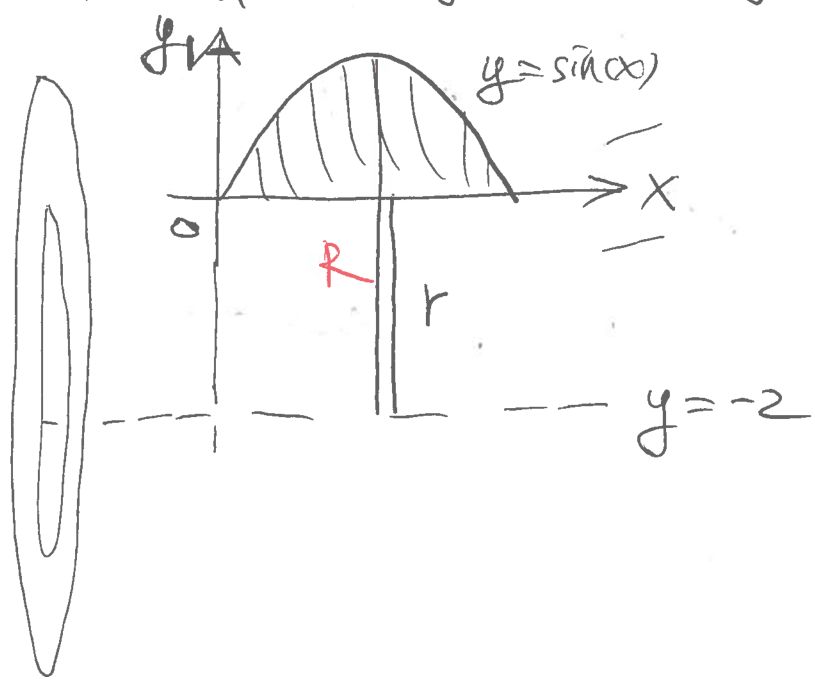
$$V_R=\pi\int_0^{16}-(8-\sqrt[4]{y})^2+(8-\frac{y}{8})^2\,dy.$$

34. Given $y=0$, $y=\sin(x)$, $0\le x\le\pi$ and find the integral of rotating volume about $y=-2$

the radius of inner circle r is 2

and the radius of outer circle R is $\sin(x)+2$. Then

$$V_R=\pi\int_0^{\pi}(\sin(x)+2)^2-2^2\,dx$$

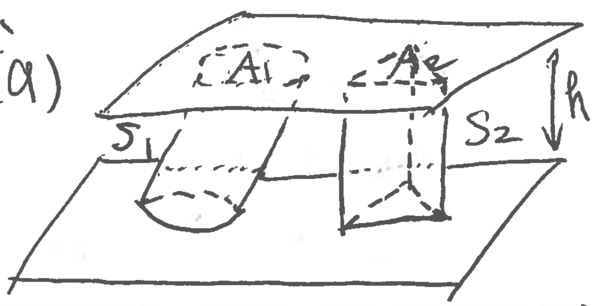


§6.2

38. Given $y = 3\sin(x^2)$ and $y = e^{\frac{x}{2}} + e^{-2x}$

65.

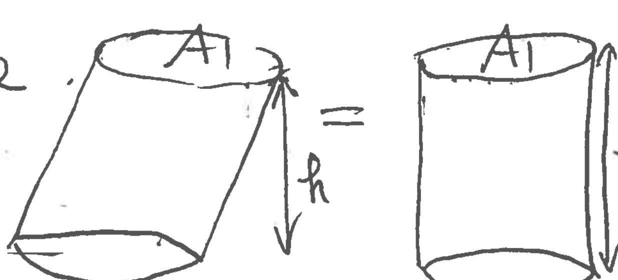
(a)



Let A_1, A_2 be the cross-sectional areas of S_1, S_2, respectively, and h be the distance between two parallel planes, then

$$\underbrace{V_{S_1}}_{\text{the volume of } S_1} = A_1 \cdot h \underset{A_1 = A_2}{=} A_2 h = \underbrace{V_{S_2}}_{\text{the volume of } S_2}$$

(b) Since

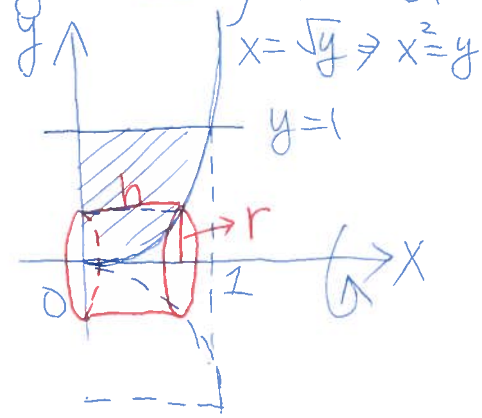


by Cavalieri's Rule.

We have the volume $= \pi r^2 h$.

§6.3

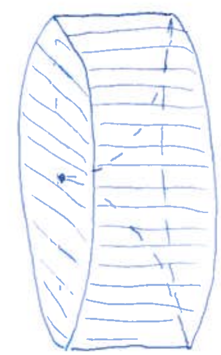
10. Given $x=\sqrt{y}$, $x=0$, $y=1$ and find the rotating volume by x-axis.



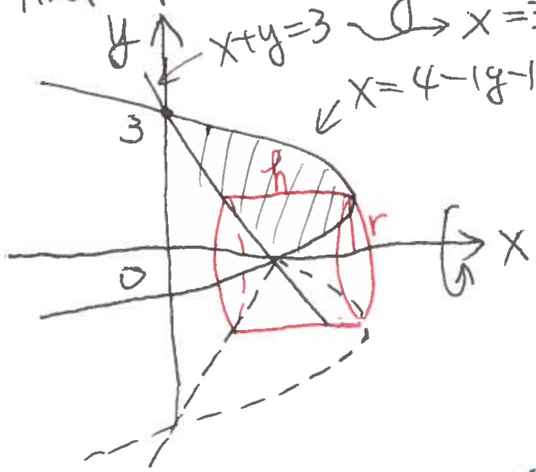
the radius r is y and the height h is $\sqrt{y}$.

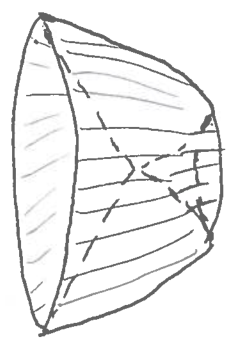
$y \in (0,1)$

$$V_r = 2\pi \int_0^1 y \cdot \sqrt{y}\, dx = 2\pi \frac{2}{5} y^{\frac{5}{2}} \Big|_0^1 = \frac{4\pi}{5}$$



14. Given $x+y=3$ and $x=4-(y-1)^2$ and find the rotating volume by x-axis.





$y \in (0,3)$

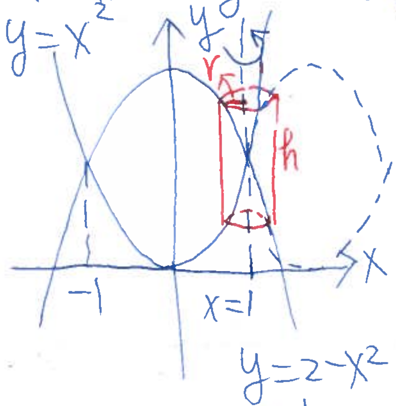
the radius r is y and the height h is $4-(y-1)^2-(3-y)$

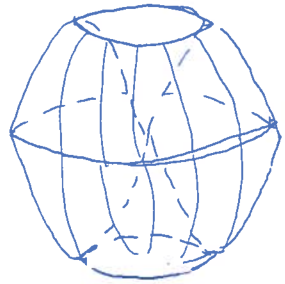
$$V_r = 2\pi \int_0^3 y\left(4-(y-1)^2-(3-y)\right)dy = 2\pi \int_0^3 y(\cancel{4}-y^2+2y\cancel{-1}\cancel{-3}+y)\,dy$$

$$= 2\pi \int_0^3 3y^2 - y^3\, dy = 2\pi \left[y^3 - \frac{y^4}{4}\right]_0^3 = 2\pi\left[27 - \frac{3}{4}\cdot 27\right] = \frac{27}{2}\pi.$$

§6.3

18. Given $y=x^2$, $y=2-x^2$ and find the rotating volume about $x=1$.





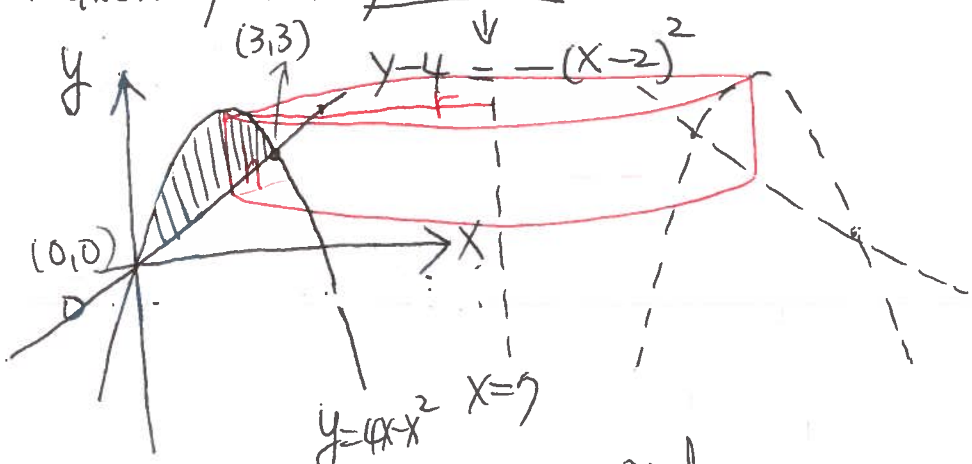
the radius r is $1-x$ and the height h is $2-x^2-x^2$ and $x \in (-1,1)$. Then

$$V_R = 2\pi \int_{-1}^{1} (1-x)(2-2x^2)\,dx = 2\pi \int_{-1}^{1} 2-2x-2x^2+2x^3\,dx$$

$$= 2\pi \left[2x - x^2 - \frac{2}{3}x^3 + \frac{x^4}{2} \right]_{-1}^{1} = 2\pi \left[2(1-(-1)) - \cancel{(1^2-(-1)^2)}^{0} - \frac{2}{3}(1^3-(-1)^3) + \cancel{\frac{1}{2}(1^4-(-1)^4)}^{0} \right]$$

$$= 2\pi \left[4 - \frac{4}{3} \right] = \frac{16}{3}\pi.$$

22. Given $y=x$, $\underline{y=4x-x^2}$ and find the integral of the rotating volume about $x=7$.



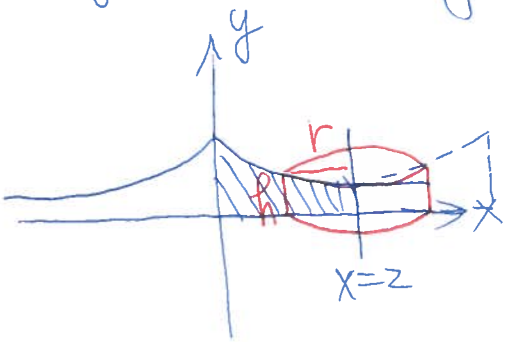
$r = 7-x$

$h = 4x-x^2-x = 3x-x^2$.

$x \in (0,3)$.

$$V_R = 2\pi \int_0^3 (7-x)(3x-x^2)\,dx.$$

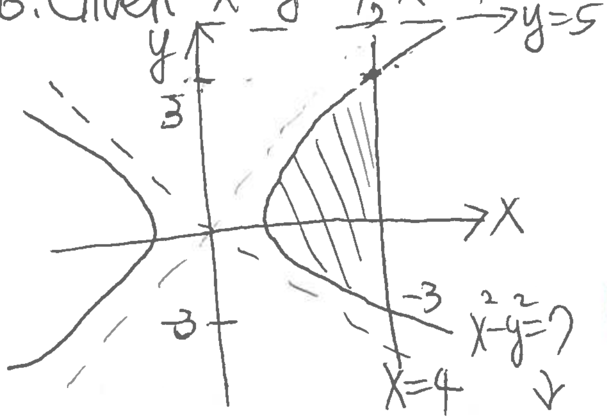
24. Given $y=\frac{1}{(1+x^2)}$, $y=0$, $x=0$, $x=2$, and find the integral for the rotating volume about $x=2$.



Then $r=2-x$, $h=\frac{1}{(1+x^2)}$ $x\in(0,2)$.

$$V_R=2\pi\int_0^2 (2-x)\frac{1}{(1+x^2)}\,dx.$$

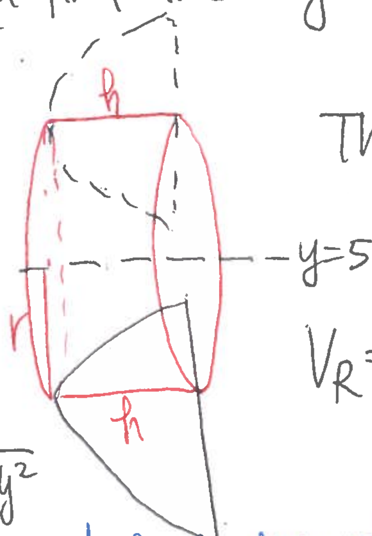
26. Given $x^2-y^2=7$, $x=4$ and find the integral for the rotating volume about $y=5$.



$x^2-y^2=7$

$\downarrow$

$x=\sqrt{7+y^2}$



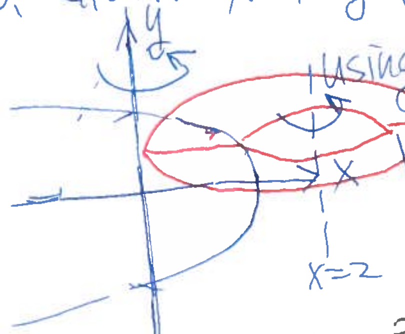
Then $r=5-y$

$h=4-\sqrt{7+y^2}$, $y\in(-3,3)$

$$V_R=2\pi\int_{-3}^{3}(5-y)(4-\sqrt{7+y^2})\,dy$$

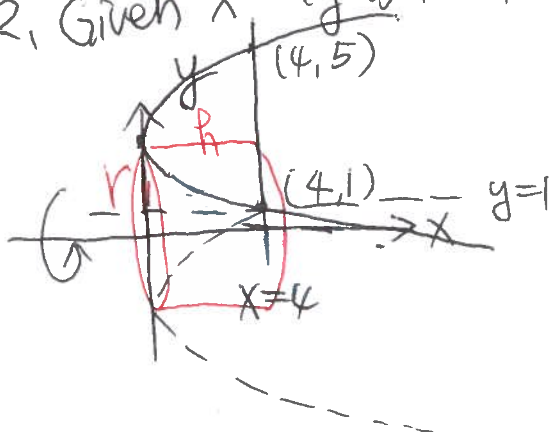
40. Given $x=1-y^4$, $x=0$ and find the rotating volume about $x=2$.

Using washer method; $r=2-(1-y^4)$, $R=2$ $y\in(-1,1)$



$$V_R=\pi\int_{-1}^{1}4-\underbrace{(2-(1-y^4))^2}_{1+2y^4+y^8}\,dy=\pi\left[3y-\frac{2}{5}y^5+\frac{y^9}{9}\right]_{-1}^{1}=\pi\frac{224}{45}.$$

42. Given $x=(y-3)^2$, $x=4$ and find the rotating volume about $y=1$



$r=y-1$, $h=4-(y-3)^2$ $y\in(1,5)$.

$$V_R=2\pi\int_1^5(y-1)(4-(y-3)^2)\,dy$$

$$=2\pi\int_1^5(y-1)[-y^2+6y-5]\,dy$$

$$=2\pi\int_1^5 -y^3+7y^2-11y+5\,dy$$

$$=2\pi\left[-\frac{y^4}{4}+\frac{7}{3}y^3-\frac{11}{2}y^2+5y\right]_1^5=\frac{128}{3}\pi$$