Honors Calculus, Math 1450 - HW5 Solutions

(1) Please see the keys of Practice 5 Q3 & Q4

(2) To solve $\int_0^a x\,dx$, by (1)(i) and Riemann sums,

let the partition of $[0,a]$ be $\{0, \frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \dots, \frac{na}{n}=a\}$ we have

$$\int_0^a x\,dx = \lim_{n\to\infty}\sum_{j=1}^{n}\frac{ja}{n}\cdot\frac{a}{n} = \lim_{n\to\infty}\frac{a^2}{n^2}\sum_{j=1}^{n} j = \lim_{n\to\infty}\frac{a^2}{n^2}\cdot\frac{(n)(n+1)}{2} = \frac{a^2}{2}.$$

Similarly, using the same partition, we have

$$\int_0^a x^2\,dx = \lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{ja}{n}\right)^2\cdot\frac{a}{n} = \lim_{n\to\infty}\frac{a^3}{n^3}\sum_{j=1}^{n} j^2 = \lim_{n\to\infty}\frac{a^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{a^3}{3}.$$

(3) §5.2

52. let $f(x)=\sqrt{1+x^2}$ and $g(x)=\sqrt{1+x}$ on $[0,1]$.

Since $f(x)\geq g(x)$ for $x\in[0,1]$. Then we have

$$\int_0^1\sqrt{1+x^2}\,dx \geq \int_0^1\sqrt{1+x}\,dx$$

53. let $f(x)=\sqrt{1+x^2}$ on $[-1,1]$. since, as $x\in[-1,1]$, $1\leq f(x)\leq\sqrt{2}$, Then

$$\int_{-1}^1 1\,dx \leq \int_{-1}^1 f(x)\,dx \leq \int_{-1}^1\sqrt{2}\,dx \Rightarrow 2\leq\int_{-1}^1\sqrt{1+x^2}\,dx\leq 2\sqrt{2}.$$

54. let $f(x)=\cos(x)$ on $[\frac{\pi}{6},\frac{\pi}{4}]$.

Since, for $x\in[\frac{\pi}{6},\frac{\pi}{4}]$, $\frac{\sqrt{2}}{2}\leq\cos(x)\leq\frac{\sqrt{3}}{2}$. Then we have

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sqrt{2}}{2} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos(x)\,dx \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sqrt{3}}{2}\,dx$$

$$\Rightarrow \frac{\sqrt{2}}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\leq\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos(x)\,dx\leq\frac{\sqrt{3}}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\Rightarrow\frac{\sqrt{2}}{24}\pi\leq\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos(x)\,dx\leq\frac{\sqrt{3}}{24}\pi$$

(4) §5.3

24. $\int_1^8 \sqrt[3]{x}\,dx = \int_1^8 x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}}\Big|_1^8 = \frac{3}{4}\left(8^{\frac{4}{3}} - 1^{\frac{4}{3}}\right) = \frac{3}{4}\left[2^4 - 1\right] = \frac{45}{4}$

30. $\int_0^2 (y-1)(2y+1)\,dy = \int_0^2 (2y^2 - y - 1)\,dy = \left[\frac{2}{3}y^3 - \frac{y^2}{2} - y\right]_0^2$

$= \left[\frac{2}{3}\cdot 2^3 - \frac{2^2}{2} - 2\right] - \left[0 - 0 - 0\right] = \frac{4}{3}$

36. $\int_0^1 10^x\,dx = \frac{1}{\ln 10}\, 10^x \Big|_0^1 = \frac{1}{\ln 10}\left[10^1 - 10^0\right] = \frac{9}{\ln 10}$

$\boxed{(10^x)' = (\ln 10)\, 10^x \Rightarrow \int 10^x\,dx = \frac{10^x}{\ln 10} + C}$

$x = \arctan(1) \Rightarrow \tan(x) = 1 \Rightarrow x = \frac{\pi}{4}$

38. $\int_0^1 \frac{4}{t^2+1}\,dt = 4\cdot \arctan(t)\Big|_0^1 = 4\cdot\left[\arctan(1) - \arctan(0)\right]$

$= 4\cdot\left[\frac{\pi}{4} - 0\right] = \pi.$

$\boxed{(\arctan(t))' = \frac{1}{t^2+1} \Rightarrow \int \frac{dt}{t^2+1} = \arctan(t) + C}$

40. $\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left[\frac{4}{u^3} + \frac{1}{u}\right] du = \left[-\frac{2}{u^2} + \ln(u)\right]_1^2$

$= \left[-\frac{2}{2^2} + \ln(2)\right] - \left[-\frac{2}{1^2} + \underbrace{\ln(1)}_{0}\right] = \frac{3}{2} + \ln(2)$

42. $\int_{-2}^2 f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2. \end{cases}$

$= \int_{-2}^0 2\,dx + \int_0^2 4 - x^2\,dx = \left[2x\right]_{-2}^0 + \left[4x - \frac{x^3}{3}\right]_0^2$

$= 0 - (2\cdot(-2)) + \left[4\cdot 2 - \frac{2^3}{3}\right] - \left[0 - 0\right] = 12 - \frac{8}{3} = \frac{28}{3}.$

54. Let $g(x) = \int_{\tan(x)}^{x^2} \frac{dt}{\sqrt{2+t^4}}$. by Fundamental Thm of Calculus, we have

$g'(x) = \frac{1}{\sqrt{2+(x^2)^4}}\cdot(x^2)' - \frac{1}{\sqrt{2+(\tan(x))^4}}\cdot(\tan(x))' = \frac{2x}{\sqrt{2+x^8}} - \frac{\sec^2(x)}{\sqrt{2+\tan^4(x)}}$

56. let $y=\int_{\cos(x)}^{5x}\cos(u^2)\,du$, by Fundamental Thm of Calculus.

We obtain

$$y' = \cos((5x)^2)\cdot(5x)' - \cos((\cos(x))^2)\cdot(\cos(x))'$$
$$= 5\cos(25x^2) + \sin(x)\cdot\cos(\cos^2(x)).$$

60. let $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \Rightarrow \int_0^x e^{-t^2}dt=\frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$

(a) $\int_a^b e^{-t^2}dt=\int_0^b e^{-t^2}dt-\int_0^a e^{-t^2}dt=\frac{\sqrt{\pi}}{2}(\operatorname{erf}(b)-\operatorname{erf}(a))$

(b) To show $y=e^{x^2}\operatorname{erf}(x)$ is a solution of $y'=2xy+\frac{2}{\sqrt{\pi}}$

Check $\text{LHS}=y'=2xe^{x^2}\operatorname{erf}(x)+e^{x^2}(\operatorname{erf}(x))'$

$$\overset{*}{=}2xe^{x^2}\operatorname{erf}(x)+e^{x^2}\cdot\frac{2}{\sqrt{\pi}}e^{-x^2}=2xe^{x^2}\cdot\operatorname{erf}(x)+\frac{2}{\sqrt{\pi}}$$

by Fundamental Thm of Calculus, $(\operatorname{erf}(x))'=\frac{2}{\sqrt{\pi}}\cdot e^{-x^2}$

and

$\text{RHS}=2xe^{x^2}\operatorname{erf}(x)+\frac{2}{\sqrt{\pi}}$

$\Rightarrow \text{RHS}=\text{LHS} \Rightarrow e^{x}\operatorname{erf}(x)$ is a solution of the given differential equation.

66. $\lim_{n\to\infty}\underbrace{\frac{1}{n}}_{dx}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right) \Rightarrow$ partition of $[0,1]$: $\{0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\cdots,\frac{n}{n}=1\}$ and $f(x)=\sqrt{x}$

So

$$\int_0^1\sqrt{x}\,dx=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1=\frac{2}{3}[1-0]=\frac{2}{3}$$

7.

(a) Since $x^2 \le x$ on $[0,1]$ and cosine function is a decreasing function on $[0,1]$ so $\cos(x^2) \ge \cos(x)$ on $[0,1]$.

(b) By (a), we have

$$\int_0^{\frac{\pi}{6}} \cos(x^2)\,dx \ge \int_0^{\frac{\pi}{6}} \cos(x)\,dx = \sin(x)\Big|_0^{\frac{\pi}{6}} = \frac{1}{2} - 0 = \frac{1}{2}.$$

(5).

(i)
$$\int e^x \sin(e^x)\,dx \underset{\uparrow}{=} \int u\sin(u)\,du \underset{\uparrow}{=} -u\cos(u) + \int \cos(u)\,du$$
$$= -u\cos(u) + \sin(u) + C$$
$$= -e^x\cos(e^x) + \sin(e^x) + C$$

u-sub
let $u = e^x$
$du = e^x dx$

Integration by Parts
$g(x) = u$, $f'(x) = \sin(u)du$
$g'(x) = du$, $f(x) = -\cos(u)$

(ii)
$$\int \frac{\log x}{x}\,dx \underset{\uparrow}{=} \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\log x)^2 + C$$

u-sub
let $u = \log x$
$du = \frac{dx}{x}$

u-sub
$u = x^2$
$du = 2x\,dx$

(iii)
$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{x}{\sqrt{1-(x^2)^2}}\,dx \overset{\downarrow}{=} \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin(u) + C$$
$$= \frac{1}{2}\arcsin(x^2) + C$$

(iv)
$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

$g(x)$	$f'(x)$	sign
x^2	$\sin(x)$	+
$2x$	$-\cos(x)$	−
2	$-\sin(x)$	+
0	$\cos(x)$	−

u-sub
$u = 1-x^2$
$du = -2x\,dx$
$\frac{du}{-2} = x\,dx$

(v)
$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.$$

(5)

(vi) $\int (\log x)^2 dx \underset{\text{I.B.P}}{=} \int 1\cdot(\log x)^2 dx = x(\log x)^2 - \int 2\log x \, dx \underset{\text{I.B.P}}{=} x(\log x)^2 - 2x\log(x) + 2\int dx$

$$\boxed{\begin{array}{c} \text{I.B.P} \\ u=(\log x)^2 \quad dv = dx \\ du = \frac{2\log x}{x}dx \quad v = x \end{array}} \qquad \boxed{\begin{array}{c} \text{I.B.P} \\ u=(\log x) \quad dv = 2dx \\ du = \frac{dx}{x} \quad v = 2x \end{array}}$$

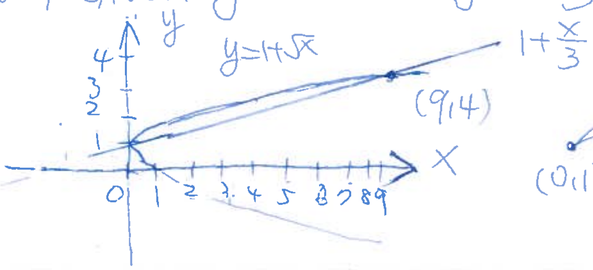
$= x(\log x)^2 - 2x\log(x) + 2x + C$.

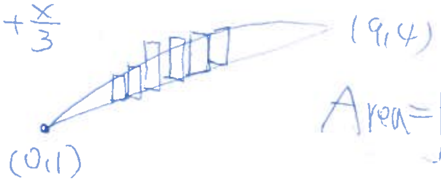
(vii) $\int \frac{dx}{x\log x} \underset{\text{u-sub}}{=} \int \frac{du}{u} = \ln(u) + c = \ln(\log(x)) + c$

$$\boxed{\begin{array}{c} \text{u-sub} \\ \text{let } u = \log x \\ du = \frac{dx}{x} \end{array}}$$

(6) §6.1

10, Given $y = 1+\sqrt{x}$, $y = \frac{3+x}{3} = 1+\frac{x}{3}$





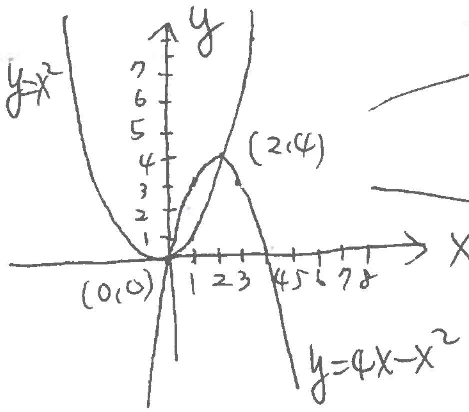
$\text{Area} = \int_0^9 \left(1+\sqrt{x} - 1 - \frac{x}{3}\right) dx$

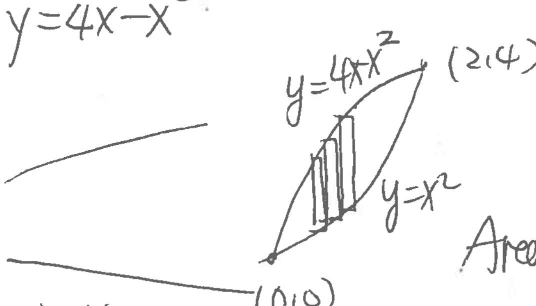
$= \int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{6}\Big|_0^9$

$= \frac{2}{3}\cdot 27 - \frac{81}{6} = 18 - \frac{27}{2}$

$= \frac{9}{2}$

12, Given $y = x^2$, $y = 4x - x^2$.



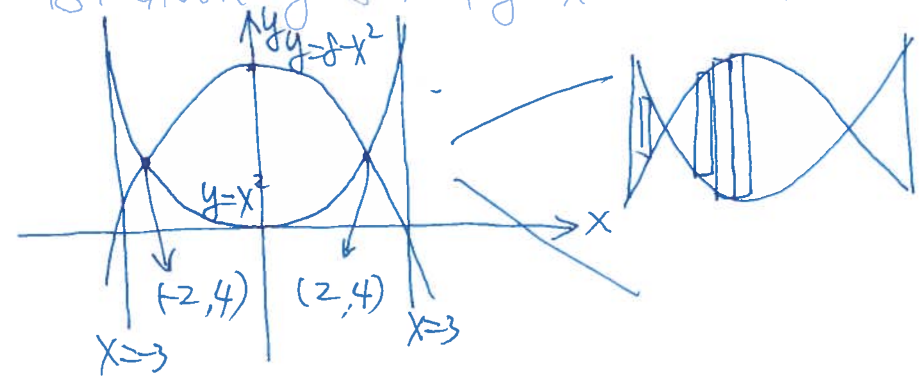


$\text{Area} = \int_0^2 4x - x^2 - x^2 \, dx$

$= 2x^2 - \frac{2}{3}x^3 \Big|_0^2 =$

$= 2\cdot 2^2 - \frac{2}{3}\cdot 2^3 = 8 - \frac{2}{3}\cdot 8 = \frac{8}{3}$.

18. Given $y=8-x^2$, $y=x^2$, $x=-3$, $x=3$

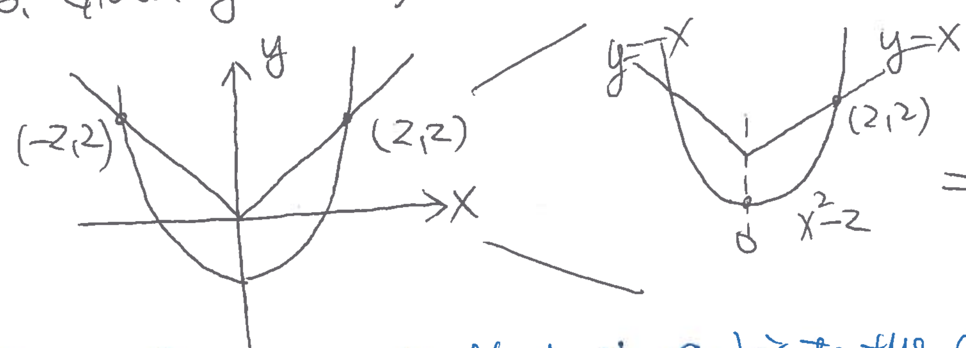


$$\text{Area} = \int_{-3}^{-2} x^2-(8-x^2)\,dx + \int_{-2}^{2} (8-x^2)-x^2\,dx + \int_{2}^{3} x^2-(8-x^2)\,dx$$

$$= 2\left[\int_0^2 8-2x^2\,dx + \int_2^3 2x^2-8\,dx\right] = 2\left\{\left[8x-\frac{2}{3}x^3\right]_0^2 + \left[\frac{2}{3}x^3-8x\right]_2^3\right\}$$

$$= 2\cdot\left\{16-\frac{2}{3}\cdot 8+\frac{2}{3}\cdot 19-8\right\} = 2\left\{8+\frac{22}{3}\right\} = \frac{92}{3}$$

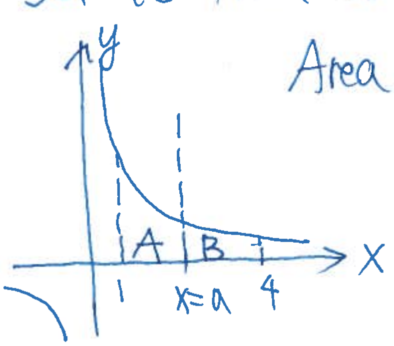
26. Given $y=|x|$, $y=x^2-2$.



$$\Rightarrow \text{Area} = 2\int_0^2 x-x^2+2\,dx$$

$$= 2\left[\frac{x^2}{2}-\frac{x^3}{3}+2x\right]_0^2 = \frac{20}{3}.$$

50. To find a such that $x=a$ bisects the area under $y=\frac{1}{x^2}$, $1\le x\le 4$.



$$\text{Area } A = \text{Area } B \iff \int_1^a \frac{dx}{x^2} = \int_a^4 \frac{dx}{x^2}$$

$$\iff -\frac{1}{x}\Big|_1^a = -\frac{1}{x}\Big|_a^4$$

$$\Rightarrow -\frac{1}{a}+1 = -\frac{1}{4}+\frac{1}{a} \Rightarrow \frac{2}{a} = \frac{5}{4} \Rightarrow a = \frac{8}{5}$$