Honors Calculus, Math 1450 – HW4 – solutions

(1)

(a) Given a function $f(x)$ which is differentiable on $\mathbb{R}$ and has two roots, says, $a$ and $b$, $a \neq b$. $f(a)=0$, $f(b)=0$

Without loss of generality, we could make $a<b$.

Then, by MVT, we obtain there is a number $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$$

So $c$ is a root of $f'$ and $f'$ has at least one root.

(b) Given a function $f(x)$ which is twice differentiable on $\mathbb{R}$ and has three roots, says, $a, b, d$, $a \neq b \neq d$. $f(a)=0$, $f(b)=0$, $f(d)=0$, W.L.O.G. we let $a<b<d$

Using the conclusions of part (a), we obtain; by MVT, there are $c, e$, $c \in (a,b)$, $e \in (b,d)$ such that

$$f'(c)=0, \quad f'(e)=0.$$

Then, by MVT, there is a number $h \in (c,e)$ such that

$$f''(h) = \frac{f'(e)-f'(c)}{e-c} = \frac{0-0}{e-c} = 0 \Rightarrow f''(h)=0$$

So $h$ is a root of $f''$ and $f''$ has at least one root.

(2) Given $f(x) = \sin(x) + \cos(x)$ on $[0, \frac{\pi}{3}]$.

To find abs. Max and min of $f(x)$, we consider $f'(x)$ first:

$$f'(x) = \cos(x) - \sin(x) = 0 \Rightarrow \cos(x) = \sin(x)$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \quad (\text{but } \frac{5\pi}{4} \notin [0, \frac{\pi}{3}]) \Rightarrow x = \frac{\pi}{4}.$$

$$f(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4}) = \sqrt{2}$$

Check endpoints: $f(0) = \sin(0) + \cos(0) = 1$

$f(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) + \cos(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{1}{2}$

Then the abs. Max is $f(\frac{\pi}{4}) = \sqrt{2}$. and abs. min is

$f(0) = 1$,

(3) Suppose $a_1 < a_2 < \cdots < a_n$ and $f(x) = \sum_{j=1}^{n} (x - a_j)^2$

To find the minimum value of $f$, we consider $f'(x)$ first:

$$f'(x) = 2(x - a_1) + 2(x - a_2) + \cdots + 2(x - a_n) = 0$$

$$f'(x) = 2(nx - a_1 - a_2 - \cdots - a_n) = 0 \Rightarrow x = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\sum_{j=1}^{n} a_j}{n}$$

Since as $x > \frac{\sum_{j=1}^{n} a_j}{n}$, $f'(x) > 0$. and as $x < \frac{\sum_{j=1}^{n} a_j}{n}$, $f'(x) < 0$.

and $\lim_{x \to -\infty} f(x) \to \infty$, $\lim_{x \to \infty} f(x) \to \infty$,

Then, $f$ is decreasing on $(-\infty, \frac{\sum_{j=1}^{n} a_j}{n})$ and increasing on $(\frac{\sum_{j=1}^{n} a_j}{n}, \infty)$

Thus $f$ has the minimum value as $x = \frac{\sum_{j=1}^{n} a_j}{n}$ which is

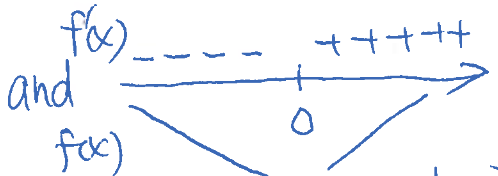
$$f\left(\frac{\sum_{j=1}^{n} a_j}{n}\right) = \sum_{j=1}^{n} \left(\frac{\sum_{j=1}^{n} a_j}{n} - a_j\right)^2$$

(4) §4.3

12. Given $f(x)=\dfrac{x^2}{x^2+3}$, the domain of $f(x)$ is $\mathbb{R}$.

(a) To find the intervals on which $f$ is increasing or decreasing, we check $f'(x)$:

$$f'(x)=\frac{2x(x^2+3)-2x^3}{(x^2+3)^2}=\frac{6x}{(x^2+3)^2}=0 \Rightarrow x=0. \quad (\text{NO DNE CASE})$$




So increasing interval is $(0,\infty)$ and decreasing interval is $(-\infty,0)$.

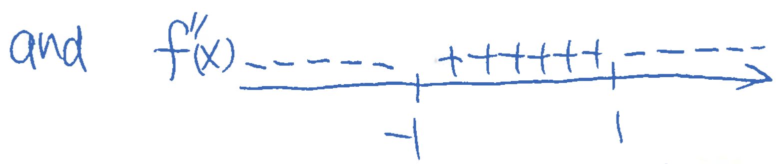
(b) Based on (a), the local extreme will be found on critical point, $x=0$, since $f'(x)<0$ as $x<0$, $f'(x)>0$ as $x>0$.

So $f(0)=0$ is a local min. and there is no local max.

(c) To find the intervals of concavity and the inflection point, we check $f''(x)$:

$$f''(x)=\frac{6(x^2+3)^2-2(x^2+3)\cdot 2x\cdot 6x}{(x^2+3)^4}=\frac{(x^2+3)(6x^2+18-24x^2)}{(x^2+3)^4}=\frac{-18(x^2+3)(x+1)(x-1)}{(x^2+3)^4}$$

$f''(x)=0 \Rightarrow x=1$ or $-1$ (We have two inflection points).




So we have concave up interval $(-1,1)$ and concave down intervals $(-\infty,-1)\cup(1,\infty)$.

(4)

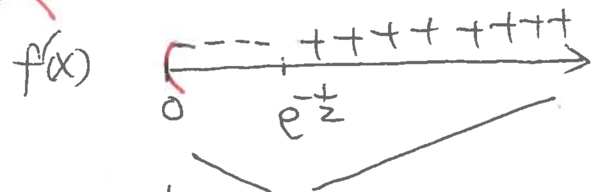
16. Given $f(x) = x^2 \ln(x)$ and the domain of $f(x)$ is $(0,\infty)$.

(a) To find the interval on which $f$ is increasing or decreasing.

We check $f'(x) = 2x\ln(x) + x^2 \cdot \frac{1}{x} = 2x\ln(x) + x = 0$. (NO DNE)

$\Rightarrow \quad x(2\ln(x)+1) = 0 \Rightarrow \cancel{x=0}$. or $e^{-\frac{1}{2}}$ ~~are two~~ a critical number~~s~~

check the number line:




So the increasing interval is $(e^{-\frac{1}{2}}, \infty)$ and the decreasing interval is $(0, e^{-\frac{1}{2}})$.

(b) Based on (a), as $x = e^{-\frac{1}{2}}$ $f(x)$ has a local min.

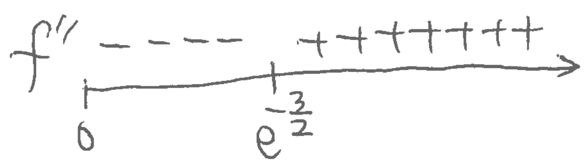
which is $f(e^{-\frac{1}{2}}) = \frac{1}{e}\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2e}$.

(c) To find the concavity intervals and inflection point of $f$.

We check $f''(x) = 2x\cdot\frac{1}{x} + 2\ln(x) + 1 = 3 + 2\ln(x) = 0$

$\Rightarrow \ln(x) = -\frac{3}{2}$, $x = e^{-\frac{3}{2}}$ is a inflection point of $f$.

check the number line: $y = e^{-3}\cdot\left(-\frac{3}{2}\right) = -\frac{3}{2e^3}$




So concave up interval is $(e^{-\frac{3}{2}}, \infty)$ and concave down interval is $(0, e^{-\frac{3}{2}})$.

(4)

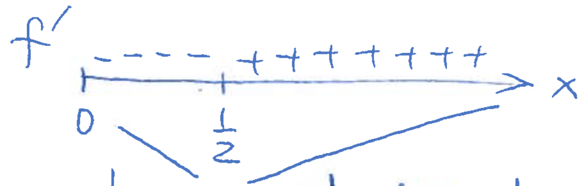
18. Given $f(x)=\sqrt{x}\,e^{-x}$ and the domain of $f(x)$ is $[0,\infty)$.

(a) To find the interval on which $f$ is increasing or decreasing.

We check $f'(x)=\frac{1}{2\sqrt{x}}e^{-x}-\sqrt{x}\,e^{-x}=\left(\frac{1-2x}{2\sqrt{x}}\right)e^{-x}=0$

$$\Rightarrow \begin{cases} f'(x)=0 \Rightarrow x=\frac{1}{2} \\ f'(x) \text{ DNE} \Rightarrow x=0 \end{cases}$$ we have two critical pts. $x=0$ & $\frac{1}{2}$.

Checking the number line:




So the increasing interval is $(\frac{1}{2},\infty)$, and the decreasing interval is $(0,\frac{1}{2})$.

(b). Based on (a), as $x=\frac{1}{2}$ $f(x)$ has a local min, which is $f(\frac{1}{2})=\sqrt{\frac{1}{2}}\,e^{-\frac{1}{2}}$.

(c) Check $f''(x)=\frac{-4\sqrt{x}-\frac{1}{\sqrt{x}}(1-2x)}{4x}e^{-x}-\left(\frac{1-2x}{2\sqrt{x}}\right)e^{-x}$

$$=\left(\frac{-4\sqrt{x}-\frac{1}{\sqrt{x}}+\cancel{\frac{2x}{\sqrt{x}}}\cancel{-2\sqrt{x}}+4x\sqrt{x}}{4x}\right)e^{-x}$$

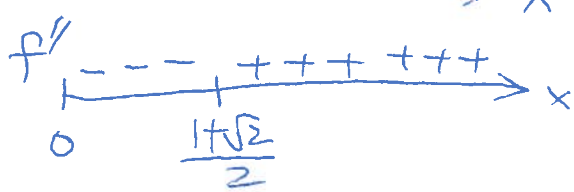
$$=\left(\frac{-4x-1+4x^2}{4x\sqrt{x}}\right)e^{-x}$$

$$\Rightarrow \begin{cases} f''(x)=0 \Leftrightarrow 4x^2-4x-1=0 \Rightarrow x=\frac{1\pm\sqrt{2}}{2} \quad (x>0) \\ f''(x) \text{ DNE} \Rightarrow x=0 \end{cases}$$

Domain

$\Rightarrow x=\frac{1+\sqrt{2}}{2}$

Checking the number line




So Concave up interval is $(\frac{1+\sqrt{2}}{2},\infty)$ and concave down interval is $(0,\frac{1+\sqrt{2}}{2})$.

and two inflection points are $x=0,\ \frac{1+\sqrt{2}}{2}$.

(4)

critical pt.

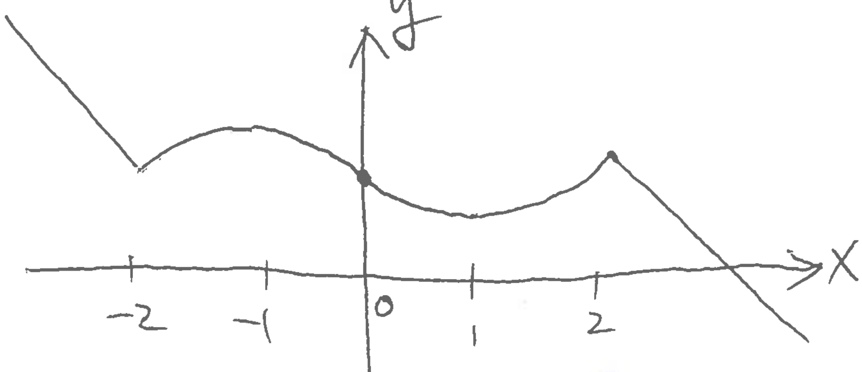
26. Given $f'(1)=f'(-1)=0$, inflection $x=0$.

$f'(x)<0$ if $|x|<1 \Rightarrow f$ is decreasing on $(-1,1)$

$f'(x)>0$ if $1<|x|<2 \Rightarrow f$ is increasing on $(-2,-1)$ and $(1,2)$.

$f'(x)=-1$ if $|x|>2 \Rightarrow f$ is a line with slope $-1$ on $(2,\infty)$ and $(-\infty,2)$

$f''(x)<0$ if $-2<x<0 \Rightarrow f$ is concave down on $(-2,0)$




66 Given $y=\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ and $f(x)=e^{\frac{-x^2}{2\sigma^2}}$.

and domain of $f$ is $\mathbb{R}$.

(a) Vertical asymptote $(x\to\infty)$, $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}e^{\frac{-x^2}{2\sigma^2}}=\lim\limits_{x\to\infty}\dfrac{1}{e^{\frac{x^2}{2\sigma^2}}}=0$.

Horizontal asymptoto $(y\to\infty)$. there is no Horizontal asym.

Maximum Value, check $f'(x)=\dfrac{-2x}{2\sigma^2}e^{\frac{-x^2}{2\sigma^2}}$.

$$\begin{cases} f'(x)=0 \Rightarrow x=0 \\ f'(x)\text{ DNE } \times \end{cases} \Rightarrow x=0 \text{ is a critical point.}$$

check number line $f'(x)$ + + + + + + | − − − − − $x$, 0

$\Rightarrow$ maximum value is $f(0)=e^0=1$.

Inflection point(s) of $f$: check $f''(x)=\dfrac{-2}{2\sigma^2}e^{\frac{-x^2}{2\sigma^2}}+\left(\dfrac{-2x}{2\sigma^2}\right)^2 e^{\frac{-x^2}{2\sigma^2}}$

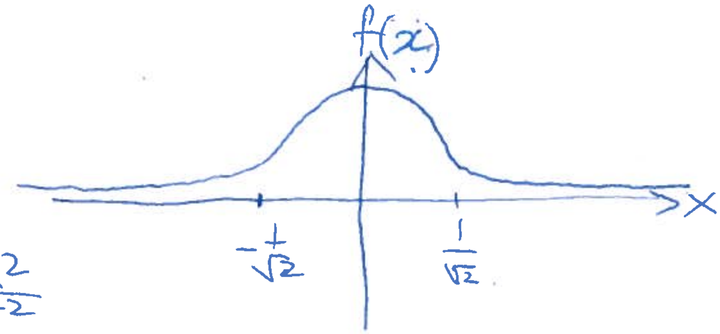
$=\dfrac{1}{\sigma^2}(\sqrt{2}x+1)(\sqrt{2}x-1)\,e^{\frac{-x^2}{2\sigma^2}}$

66.

(a) (conti.)

So $f''(x)=0 \Rightarrow x=\frac{1}{\sqrt{2}}$ or $\frac{-1}{\sqrt{2}}$ are two inflection points.

(NO DNE CASE)




(b) Since $f'(x)=\frac{-2x}{2\sigma^2}e^{\frac{-x^2}{2\sigma^2}}$

If $\sigma$ gets smaller, the rate of change of f will get bigger.
So f will change rapidly and the shape of f will get flat.
On the other hand, if $\sigma$ gets larger, the rate of change of f will get smaller and the shape of f will get sharp in the central part.

68. Given $f(x)=axe^{bx^2}$. To find a, b such that f has maximum value 1 at $x=2$.

First, we check $f'(x)=(a+2abx^2)e^{bx^2}$

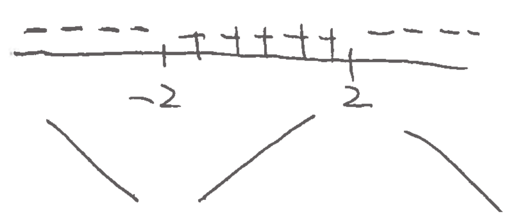
$f'(x)=0 \Rightarrow a+2abx^2=0 \Rightarrow x^2=-\frac{a}{2ab}=-\frac{1}{2b}$

($e^{bx^2}>0$ for all $x\in\mathbb{R}$)

Since, as $x=2$, f has max. value $\Rightarrow 2^2=-\frac{1}{2b} \Rightarrow b=-\frac{1}{8}$.

Since $f(2)=1$ and $b=-\frac{1}{8}$, $\Rightarrow 1=f(2)=2ae^{-\frac{1}{8}\cdot 4} \Rightarrow a=\frac{\sqrt{e}}{2}$

(check. number line if $a=\frac{\sqrt{e}}{2}$ and $b=-\frac{1}{8}$:




$\Rightarrow$ f has local max @ $x=2$)

(4)

76. (a) let $f(x)=e^x$, $g(x)=1+x$, $\forall x \geq 0$

Since $f'(x)=e^x$, $g'(x)=1$. $e^x \geq 1$ $\forall x \geq 0$

$\Rightarrow$ $f$ increases faster than $g(x)$

and $f(0)=e^0=1$, $g(0)=1$ $\Rightarrow$ $f$ and $g$ has the same value

as $x=0$, so $f$ is also larger than $g$ for all $x \geq 0$

$\Rightarrow$ $e^x \geq 1+x$ $\forall x \geq 0$

(b) let $h(x)=1+x+\frac{x^2}{2}$ for $x \geq 0$, Since $h'(x)=1+x$

and by part (a), we know $e^x \geq 1+x$ $\forall x \geq 0$.

so $f'(x) \geq h'(x)$ $\forall x \geq 0$.

Similarly, $f$ is growing faster than $h$ and $f(0)=1=h(0)$,

so $f(x) \geq h(x)$ $\Rightarrow$ $e^x \geq 1+x+\frac{x^2}{2}$

(c) To show $e^x \geq 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$ is true for all $n \geq 1$, $x \geq 0$.

First, as $n=1$, we have

LHS $=e^x$ and RHS $=1+x$, then, by part (a),

LHS $\geq$ RHS, this statement is true as $n=1$.

Then, assume, as $n=k$,

$$e^x \geq 1+x+\frac{x^2}{2!}+\cdots+\frac{x^k}{(k)!}$$

P.8

(4)
76 (conti.)

So, as $n=k+1$, we have

$LHS = e^x$ and $(LHS)' = e^x$.

$RHS = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^k}{k!}+\frac{x^{k+1}}{(k+1)!}$, $(RHS)' = 1+\frac{2x}{2!}+\cdots+\frac{kx^{k-1}}{k!}+\frac{(k+1)x^k}{(k+1)!}$

$= 1+x+\cdots+\frac{x^{k-1}}{(k-1)!}+\frac{x^k}{k!}$

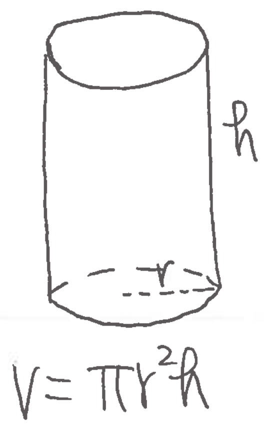
Since $e^x \geq 1+x+\frac{x^2}{2!}+\cdots+\frac{x^k}{k!} \Rightarrow (LHS)' \geq (RHS)'$. which means LHS is growing faster that RHS. and

Since LHS $= 1 =$ RHS as $x=0$. So LHS $\geq$ RHS $\forall x \geq 0$.

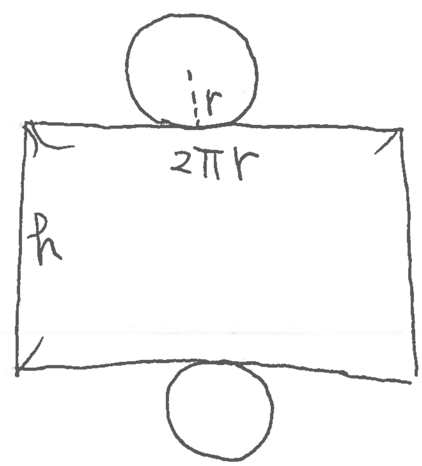
$\Rightarrow$ the statement is true as $n=k+1$.

Thus, by math induction, the statement is true $\forall n \in \mathbb{N}$.

(5).




$\Rightarrow$




$S = 2\pi r h + 2\pi r^2$

let the height of cylinder be $h$, the radius of circle be $r$.

we have the volume of circular cylinder: $V = \pi r^2 h$

and the area of surface $S = 2\pi r h + 2 \cdot \pi r^2$

Given $V=1$, To find the min. value of $S$.

Since $1 = V = \pi r^2 h \Rightarrow h = \frac{1}{\pi r^2}$. put this relation into $S$,

we obtain $S = 2\pi r h + 2\pi r^2 = \frac{2\pi r}{\pi r^2} + 2\pi r^2 = \frac{2}{r} + 2\pi r^2$.

Now $S$ is a function of $r$ and the domain of $S(r)$ is $(0, \infty)$.

$$S(r) = \frac{2 + 2\pi r^3}{r}, \quad S'(r) = \frac{6\pi r^3 - 2 - 2\pi r^3}{r^2} = \frac{4\pi r^3 - 2}{r^2}$$

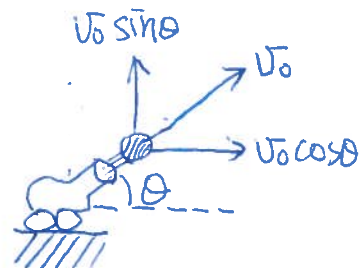
$S'(r) = 0 \Rightarrow 4\pi r^3 - 2 = 0 \Rightarrow r = \sqrt[3]{\frac{1}{2\pi}}$ (two critical points).

$S'(r)$ DNE $\Rightarrow r = 0$ ($\times$ since $r \in (0,\infty)$)

Check the number line: $S'(r)$   $0$ - - - - $\sqrt[3]{\frac{1}{2\pi}}$ + + + + + + $\rightarrow r$

so, as $r = \sqrt[3]{\frac{1}{2\pi}}$, $h = \frac{\sqrt[3]{4\pi^2}}{\pi}$, $S$ has min. value.

(6) Given the height of projectile be $y(t) = -16t^2 + (v_0 \sin\theta)t$




$\theta \in [0, \frac{\pi}{2}]$

(a) On the horizontal direction, the position of projectile is $x(t) = (v_0 \cos\theta)t$.

To find the relation between $x$ and $y$, we have

$t = \frac{x(t)}{v_0 \cos\theta}$, put this into $y$, we obtain

$$y(t) = -\frac{16x^2(t)}{v_0^2 \cos^2\theta} + \frac{v_0 \sin\theta}{v_0 \cos\theta} x(t) \Rightarrow y = -\frac{16}{v_0^2 \cos^2\theta} x^2 + (\tan\theta) x$$

which is a parabola.

(6) (conti.)

(b) Find the max. value of $X(t) = (v_0 \cos\theta)t$. the farest place this projectile can be is happened as $y(t) = 0$. which is

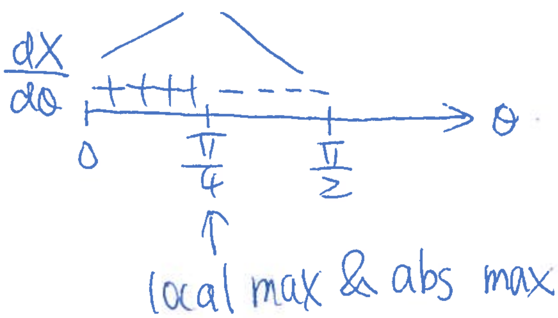
$$-16t^2 + (v_0 \sin\theta)t = 0 \Rightarrow t = 0 \text{ or } t = \frac{v_0 \sin\theta}{16}$$

Since $t = 0$ is the initial point, so we only consider $t = \frac{v_0 \sin\theta}{16}$.

So $X\left(\frac{v_0 \sin\theta}{16}\right) = \frac{v_0^2 \sin\theta \cos\theta}{16} = \frac{v_0^2 \sin 2\theta}{32}$ and now $x$ is only dependents on $\theta$.

Then, $\frac{dx}{d\theta} = \frac{2 v_0^2 \cos 2\theta}{32} = 0 \Rightarrow \cos(2\theta) = 0$, $2\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$

is a critical point. by checking the number line, we have




Thus, as $\theta = \frac{\pi}{4}$. $X$ has max. value ...

(7) let $f(x) = V(x) + \frac{1}{2} m \dot{x}^2$

(a) To show $V(x(t)) + \frac{1}{2} m \dot{x}^2(t)$ remains constant, it is sufficiently to prove $\frac{df(x)}{dt} = 0$. (since $x$ is dependent on $t$, so we should consider $\frac{d}{dt}$, not $\frac{d}{dx}$)

Since $\frac{df(x)}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} + \frac{1}{2} m \cdot 2 \dot{x} \cdot \ddot{x}$

$$= \frac{dV}{dx} \cdot \dot{x} + m \dot{x} \ddot{x} \quad \text{and} \quad \frac{dV}{dx} = -m\ddot{x}$$

we have $\frac{df(x(t))}{dt} = -m\ddot{x} \cdot \dot{x} + m\dot{x}\ddot{x} = 0$.

(7) (conti.)

(b). Given $m\ddot{x} = -kx$ where $m=1$, $k=2$, $x(0)=0$, $\dot{x}(0)=4$.

By (a) we have $V(x) + \frac{1}{2}m\dot{x}^2$ is a constant. and $\frac{dV}{dx} = kx$
$\Rightarrow V = \frac{kx^2}{2}$

So, based on the givens, as $t=0$,

we have $\frac{k}{2}x^2(0) + \frac{1}{2}m\dot{x}(0)^2 = 2\cdot 0 + \frac{1}{2}\cdot 1\cdot 16 = 8$

$$\Rightarrow \quad x^2 + \frac{1}{2}\cdot 1\cdot \dot{x}^2 = 8$$

As $\dot{x}(t)=0$ we have the maximum distance, then

$$x^2(t) + \frac{1}{2}\cdot 1\cdot 0 = 8 \Rightarrow x(t) = 2\sqrt{2}.$$

(8) 4.4

8. $\lim_{x\to 1}\frac{x^a-1}{x^b-1} \overset{(\frac{0}{0})}{\underset{L'}{=}} \lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

10. $\lim_{x\to 0}\frac{\sin(4x)}{\tan(5x)} \overset{(\frac{0}{0})}{\underset{L'}{=}} \lim_{x\to 0}\frac{4\cos(4x)}{5\sec^2(5x)} = \frac{4}{5}$

20. $\lim_{x\to 1}\frac{\ln(x)}{\sin(\pi x)} \overset{(\frac{0}{0})}{\underset{L'}{=}} \lim_{x\to 1}\frac{\frac{1}{x}}{\pi\cos(\pi x)} = \frac{1}{\pi}$

28. $\lim_{x\to\infty}\frac{(\ln(x))^2}{x} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}\frac{2\frac{\ln(x)}{x}}{1} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}\frac{2}{x} = 0$

40. $\lim_{x\to-\infty} x^2e^x \overset{(\infty\cdot 0)}{\underset{(L')}{=}} \lim_{x\to-\infty}\frac{x^2}{e^{-x}} \overset{(\frac{\infty}{\infty})}{\underset{(L')}{=}} \lim_{x\to-\infty}\frac{2x}{-e^{-x}} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to-\infty}\frac{2}{e^{-x}} \Rightarrow$ DNE

42. $\lim_{x\to 0^+}\sin(x)\cdot\ln(x) \Rightarrow$ DNE

56. $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} \overset{1^\infty}{\underset{(L')}{=}} \lim_{x\to\infty} e^{bx\ln(1+\frac{a}{x})} = e^{\lim_{x\to\infty} bx\ln(1+\frac{a}{x})} = e^{ab}$

since $\lim_{x\to\infty} bx\ln(1+\frac{a}{x}) \overset{0\cdot\infty}{=} \lim_{x\to\infty}\frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}} \underset{(L')}{=} \lim_{x\to\infty}\frac{\frac{x}{x+a}\cdot\frac{-a}{x^2}}{\frac{-1}{bx^2}} = \lim_{x\to\infty}\frac{abx}{x+a} = ab$