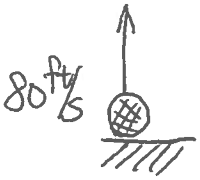
Honors Calculus, Math 1450-HW3, Solutions.

§3.7

(10) Assume the initial velocity of a ball is $80 \,\mathrm{ft/s}$ thrown vertically from ground and after t seconds the height of ball is $s(t) = 80t - 16t^2$.



(a) Maximum height $\Rightarrow$ velocity of ball is zero

$$\Rightarrow \dot{s}(t) = 0.$$

Then $\dot{s}(t) = 80 - 32t \doteq 0 \Rightarrow t = \frac{80}{32} = \frac{5}{2}$

We have the maximum height: $s(\frac{5}{2}) = 200 - 100 = 100 \,(\mathrm{ft})$.

(b) When $s(t) = 96 \,\mathrm{ft}$. we have

$$80t - 16t^2 = 96 \Rightarrow 80t - 16t^2 - 96 = 0 \Rightarrow -t^2 + 5t - 6 = 0$$

$$\Rightarrow (t-2)(t-3) = 0 \Rightarrow t = 2 \text{ or } 3.$$

By (a), we know as $t = \frac{5}{2}$, ball is on the highest position so the velocity of the ball when it is $96 \,\mathrm{ft}$ on its way up

is $\dot{s}(2) = 80 - 64 = 16 \,\mathrm{ft/s}$

and the velocity on its way down is $\dot{s}(3) = 80 - 96 = -16 \,\mathrm{ft/s}$

§3.7

(20) Given Newton's Law of Gravitation: $F = \frac{GmM}{r^2}$

(a)
$$\frac{dF}{dr} = \underset{\uparrow}{-2}\,\frac{GmM}{r^3}$$ which is the rate of change of force w.r.t r.

"$-$" means as r increases, F is decreasing.

(b) The earth attracts an object with a force that decrease at the rate of 2 N/km as $r = 20{,}000$ km.

$$\Rightarrow \left.\frac{d\vec{F}}{dr}\right|_{r=20000} = -2 \Rightarrow -2\frac{GmM}{(20000)^3} = -2. \Rightarrow GmM = (20000)^3$$

Then $$\left.\frac{dF}{dr}\right|_{r=10000} = -2\frac{GmM}{(10000)^3} = -2\frac{(20000)^3}{(10000)^3} = -2\cdot\frac{2^3}{1^3} = -16.$$

(34) A model for the rate of change of the fish population is given by

$$\frac{dP}{dt} = \overset{\text{birth rate}}{\overset{\downarrow}{r_0}}\left(1 - \frac{P(t)}{\underset{\nwarrow\text{carrying capacity.}}{P_c}}\right)P(t) - \overset{\text{harvesting rate}}{\overset{\downarrow}{\beta}}\,P(t)$$

(a)
a stable population means we have $\frac{dP}{dt} = 0$

(34)

(b) Given $P_c = 10000$, $r_0 = 5\%$, and $\beta = 4\%$.

Finding P such that $\frac{dP}{dt} = 0$, we have

$$0 = \frac{5}{100}\left(1 - \frac{P}{10000}\right)P - \frac{4}{100}P$$

$$\overset{\times 100}{\Rightarrow} 0 = 5\left(1 - \frac{P}{10000}\right)P - 4P \Rightarrow -\frac{\cancel{5}P^2}{\underset{2000}{\cancel{10000}}} - 4P + 5P = 0$$

$$\overset{\times 2000}{\Rightarrow} P^2 - 2000P = 0 \Rightarrow P(P-2000) = 0$$, so $P = 0$ or 2000

Thus the stable population level is $P = 2000$.

(c) If $\beta = 5\%$, by math calculating, we have $P^2 = 0 \Rightarrow P = 0$.

§3.8

(4) ~~Let the~~ population of bacteria is $P(t)$. and the constant relative growth rate be K. and we have

$P(2) = 600$. $P(8) = 75000$ and $P(t) = P(0)e^{Kt}$

(a) Find K, we have
$$600 = P(2) = P(0)e^{2K} \quad \text{— (1)}$$
$$75000 = P(8) = P(0)e^{8K} \quad \text{— (2)}$$

$$\frac{(2)}{(1)} \rightarrow \frac{75000}{600} = \frac{P(0)e^{8K}}{P(0)e^{2K}} \Rightarrow 125 = e^{6K}$$

taking "ln" $\cancel{\Rightarrow}$

$$\ln 125 = 6K \Rightarrow K = \frac{\ln 125}{6}$$

Then, we put K into (1). and get

$$600 = P(0) \cdot e^{2 \cdot \frac{\ln 125}{6}} \Rightarrow 600 = P(0) \cdot e^{\frac{\ln 125}{3}} = P(0) \cdot e^{\ln 125^{\frac{1}{3}}}$$
$$= P(0) \cdot (125)^{\frac{1}{3}}$$

$$\Rightarrow 600 = P(0) \cdot 5 \Rightarrow P(0) = 120.$$

§3.8

(4)(b) by (a), we have

$$p(t) = 120 \cdot e^{\frac{\ln 125}{6} t} \quad \text{or} \quad 120 \cdot (125)^{\frac{t}{6}}$$

(c) $p(5) = 120 \cdot (125)^{\frac{5}{6}}$

(d) $p(t) = 120 \cdot e^{(\ln 125)\frac{t}{6}}$, $\Rightarrow \dot{p}(t) = \overset{20}{\cancel{120}} \cdot \overbrace{(\ln 125)}^{3 \ln 5} \cdot \frac{1}{6} \cdot e^{(\ln 125) \cdot \frac{t}{6}}$

Then $\dot{p}(5) = 20 \cdot 3 \ln 5 \cdot e^{(\ln 125)\frac{5}{6}}$ or $20 \cdot 3 \cdot \ln 5 \cdot (125)^{\frac{5}{6}}$

(e) Finding t such that $p(t) = 200{,}000$, we have

$$200000 = 120\,(125)^{\frac{t}{6}} \Rightarrow \frac{\overset{1}{\cancel{2}}0000\cancel{0}}{\underset{6}{\cancel{12}}\cancel{0}} = (125)^{\frac{t}{6}}$$

$$\Rightarrow \frac{5000}{3} = (125)^{\frac{t}{6}} \xrightarrow{\text{Taking ``ln''}} \ln\left(\frac{5000}{3}\right) = \frac{t}{6} \ln(125)$$

$$\Rightarrow \frac{t}{6} = \frac{\ln\left(\frac{5000}{3}\right)}{\ln(125)} \rightarrow t = 6 \cdot \frac{\ln\left(\frac{5000}{3}\right)}{\ln(125)}$$

(11) Given the half-life of ^{14}C (carbon) be 5730 years.

Let $m(t)$ be the level of radioactivity at time t. we have

$m(t) = m(0)\, e^{-kt}$ with rate $-k$.

Then we have $\quad m(5730) = \frac{1}{2} m(0)$. $\quad \Rightarrow \frac{1}{2} = e^{-5730k}$.

$\quad\quad\quad\quad\quad\quad \| \; m(0) \cdot e^{-5730k}$

$$\xrightarrow{\text{Taking ``ln''}} \ln\left(\tfrac{1}{2}\right) = \ln\left(e^{-5730k}\right) = -5730k.$$

$$\Rightarrow -\ln 2 = -5730k \Rightarrow k = \frac{\ln 2}{5730}.$$

(11) (conti.)

Then we have $m(t)=m(0)\cdot e^{-\frac{\ln 2}{5730}t} \Rightarrow \frac{m(t)}{m(0)} = e^{-\frac{\ln 2}{5730}t}$

Now, $\frac{m(t)}{m(0)}$ of the fragment is $\frac{74}{100}$, so.

$$\frac{74}{100} = e^{-\frac{\ln 2}{5730}t} \xrightarrow{\text{Taking ``ln''}} \ln\left(\frac{74}{100}\right) = \ln\left(e^{-\frac{\ln 2}{5730}t}\right) = -\frac{\ln 2}{5730}t$$

$$\Rightarrow t = -\ln\left(\frac{74}{100}\right)\cdot\frac{5730}{\ln(2)}$$

18) If \$1000 is borrowed at 8% interest, find the amounts due

(a) at the end of 3 years if

(i) annually: $1000\cdot\left(1+\frac{0.08}{1}\right)^3 = 1000\cdot(1.08)^3$

$(A_0=1000)$

(ii) quarterly: $1000\left(1+\frac{0.08}{4}\right)^{4\cdot3} = 1000\cdot(1.02)^{12}$

(iii) monthly: $1000\left(1+\frac{0.08}{12}\right)^{12\cdot3} = 1000\left(1+\frac{0.02}{3}\right)^{36}$

(iv) weekly: $1000\left(1+\frac{0.08}{52}\right)^{52\cdot3}$ or 156

(v) daily: $1000\left(1+\frac{0.08}{365}\right)^{365\cdot3}$ or 1095

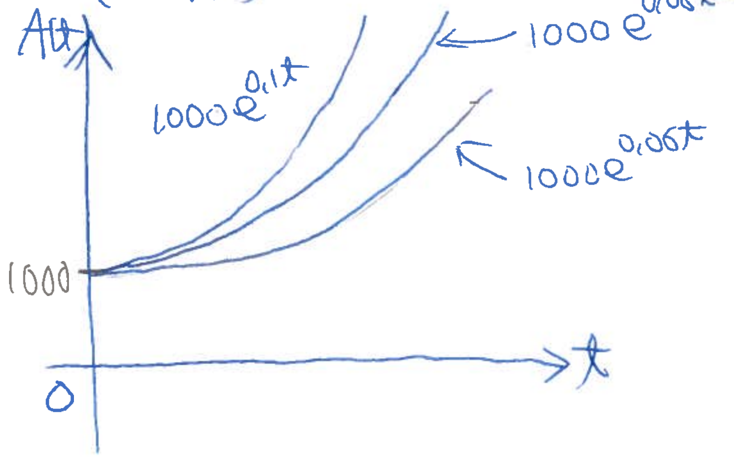
(vi) hourly: $1000\left(1+\frac{0.08}{365\cdot24}\right)^{365\cdot24\cdot3}$ or 8760·3 or 26280

(vii) continuously ↘ $A(3)=\lim_{n\to\infty}1000\cdot\left(1+\frac{0.08}{n}\right)^{nt} = 1000\cdot e^{0.08\cdot3}$

(b) For 6%, $A(t)=1000e^{0.06t}$

For 8%, $A(t)=1000e^{0.08t}$

For 10% $A(t)=1000e^{0.1t}$



§3.9

(6) let V be the volume of a sphere, r be the radius of the sphere, we have $V=\frac{4}{3}\pi r^3$.

Given $\frac{dr}{dt}=4$ mm/s, Finding $\frac{dV}{dt}$ when diameter is 80 mm

which means $r=40$mm.

Since $V=\frac{4}{3}\pi r^3$, we have $\frac{dV}{dt}=\frac{4}{3}\pi\cdot 3r^2\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}$.

Thus $\left.\frac{dV}{dt}\right|_{r=40}=4\pi(40)^2\cdot\underset{\substack{\sim\\ \uparrow\\ \frac{dr}{dt}}}{4}=25600\pi$

10. The trajectory of the particle is $y=\sqrt{1+x^3}\Rightarrow\frac{dy}{dt}=\frac{1}{2}(1+x^3)^{-\frac{1}{2}}\cdot 3x^2\frac{dx}{dt}$

As $(x,y)=(2,3)$ we have $\left.\frac{dy}{dt}\right|_{(2,3)}=4$ cm/s², Then.

$$4=\frac{1}{2}(1+2^3)^{-\frac{1}{2}}\cdot 3\cdot 4\left.\frac{dx}{dt}\right|_{(2,3)}\Rightarrow 4=2\left.\frac{dx}{dt}\right|_{(2,3)}\Rightarrow\left.\frac{dx}{dt}\right|_{(2,3)}=2\ \mathrm{cm}/s^2$$

22. Given the trajectory of particle be $y=\sqrt{x}$ and as $(x,y)=(4,2)$ $\frac{dx}{dt}=3$ cm/s. Finding the rate of change of the distance from (x,y) to $(0,0)$, we have to consider

$$s=\sqrt{x^2+y^2}\quad\text{or}\quad s^2=x^2+y^2.$$

Then, taking $\frac{d}{dt}$ on both sides, we have $2s\cdot\frac{ds}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$

P.t.

22. (conti.)

As $(x,y)=(4,2)$ we have $\frac{dy}{dt}\Big|_{(4,2)} = \frac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dt}\Big|_{(4,2)}$

$$\Rightarrow \frac{dy}{dt}\Big|_{(4,2)} = \frac{1}{2}\cdot\frac{1}{2}\cdot 3 = \frac{3}{4}$$

and $s=\sqrt{4^2+2^2}=2\sqrt{5}$

$$\Rightarrow 2s\Big|_{(4,2)}\frac{ds}{dt}\Big|_{(4,2)} = 2x\frac{dx}{dt}\Big|_{(4,2)} + 2y\frac{dy}{dt}\Big|_{(4,2)}$$

$$\Rightarrow 4\sqrt{5}\cdot\frac{ds}{dt}\Big|_{(4,2)} = 8\cdot 3 + 4\cdot\frac{3}{4} \Rightarrow \frac{ds}{dt}\Big|_{(4,2)} = \frac{27}{4\sqrt{5}} = \frac{27}{20}\sqrt{5}$$

33. Given $\frac{dR_1}{dt} = 0.3\,\Omega/s$, $\frac{dR_2}{dt} = 0.2\,\Omega/s$ and $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. —(*)

As $R_1 = 80\Omega$, $R_2 = 100\Omega$, we have $\frac{1}{R} = \frac{1}{80} + \frac{1}{100} = \frac{9}{400} \Rightarrow R = \frac{400}{9}$.

Then, ~~taking~~ "$\frac{d}{dt}$" on (*), we obtain

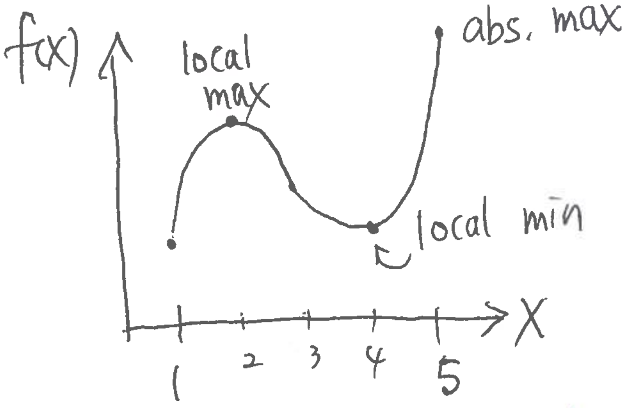
$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \Rightarrow -\left(\frac{9}{400}\right)^2\frac{dR}{dt}\Big|_{\substack{R_1=80\\R_2=100}} = -\frac{1}{80^2}\cdot 0.3 - \frac{1}{100^2}\cdot 0.2$$

$$\Rightarrow \frac{dR}{dt}\Big|_{\substack{R_1=80\\R_2=100}} = \frac{\frac{3}{64000} + \frac{2}{100000}}{\frac{81}{160000}} = \frac{107}{810}$$

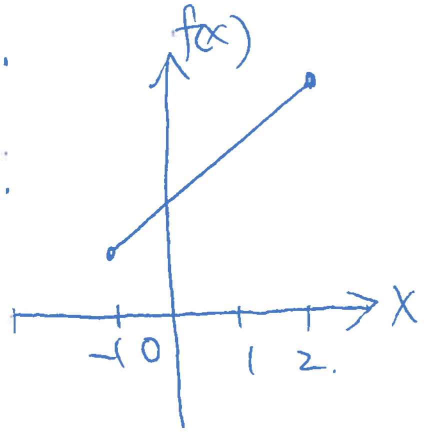
§4.1

8. Suppose f is continuous on [1,5] and has absolute min at 1, absolute max at 5, local max at 2, local min at 4.

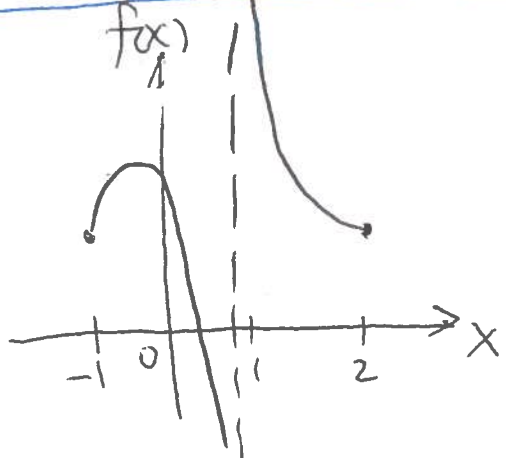
We have the graph:



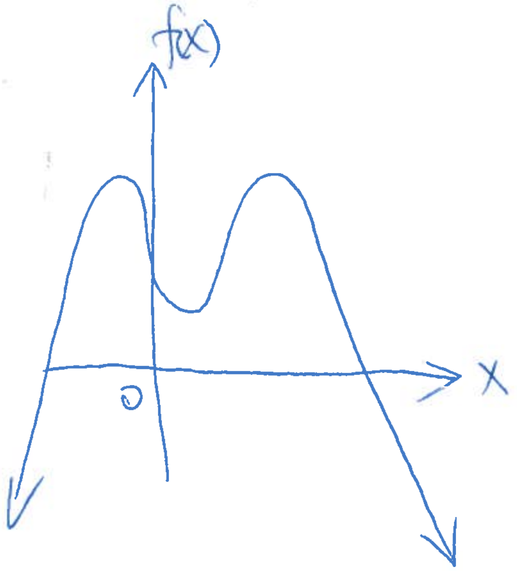
12. (a) Let f be a function has an abs. max but no local max on [-1,2]. we have the graph:



(b) Let f be a function has a local max but no abs. max we have the graph

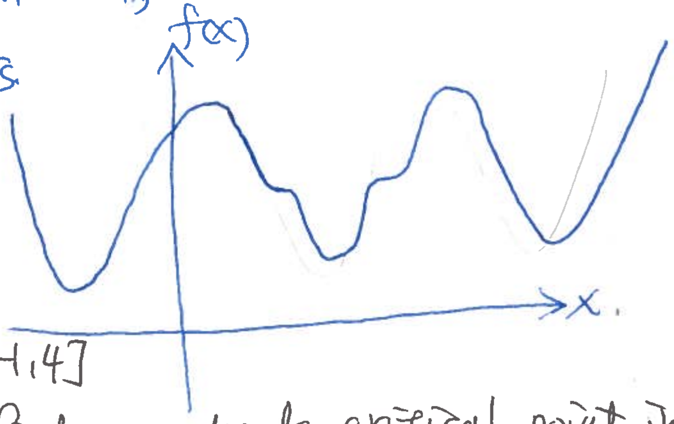


14. (a) Let f be a function has two local max. one local min, and no abs. min., we have the graph



14. (conti.)

(b) Let f be a function has 3 local mins, 2 local max. and seven critical numbers.



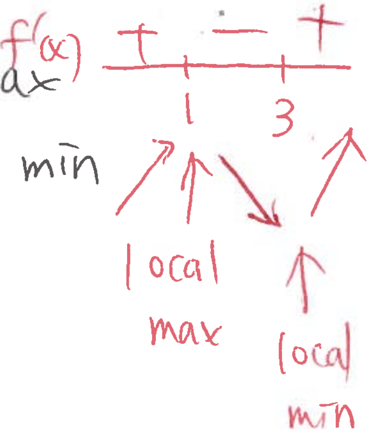
50. Given $f(x)=x^3-6x^2+9x+2$, On $[-1,4]$

To find abs. max & abs. min, first we check critical point in $[-1,4]$.

We have $f'(x)=3x^2-12x+9=0 \Rightarrow (x-1)(x-3)=0 \Rightarrow x=1$ or 3.

Then as $x=1$, $f(1)=1-6+9+2=6$ ← local max

as $x=3$, $f(3)=27-54+27+2=2$ ← local min



Then, check the endpoints $x=-1$. $x=4$.

as $x=-1$, $f(-1)=-1-6-9+2=-14$

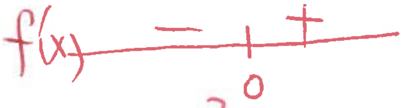
$x=4$ $f(4)=64-96+36+2=6$

$\Rightarrow$ As $x=4$ and $x=1$ $f(x)$ has abs. max 6.

As $x=-1$ $f(x)$ has abs. min -14

54. Given $f(x)=\frac{x^2-4}{x^2+4}$ on $[-4,4]$. To find abs. max & abs. min. first we check the critical point(s) in $[-4,4]$, we have.

$$f'(x)=\frac{2x(x^2+4)-2x(x^2-4)}{(x^2+4)^2}=\frac{16x}{(x^2+4)^2}=0 \Rightarrow x=0.$$



And $f(0)=-1$. $\Longrightarrow$ local min abs. min

Then check $x=-4$, $f(-4)=\frac{12}{\cancel{16}\,20}=\frac{3}{\cancel{4}\,5}$ $\Big\}\Rightarrow$ abs. max

$x=4$, $f(4)=\frac{3}{\cancel{4}\,5}$

56. let $f(t) = \sqrt[3]{t}\,(8-t)$ on $[0,8]$, For abs. min and max.

first, we check the critical point(s), we have

$$f'(t) = \frac{1}{3}t^{-\frac{2}{3}}(8-t) + \sqrt[3]{t}(-1) = \frac{8-t}{3t^{\frac{2}{3}}} - t^{\frac{1}{3}} = \frac{8-4t}{3t^{\frac{2}{3}}} = 0$$

$t=2 \Rightarrow f(2) = 6\sqrt[3]{2}$

$t=0 \Rightarrow f(0) = 0.$

(for DNE f')

Then, checking endpoint $x=8$, $f(8)=0$.

$\Rightarrow$ abs. max is $f(2) = 6\sqrt[3]{2}$ and abs. min are $f(0)=f(8)=0$

60. let $f(x) = x - \ln x$ on $[\frac{1}{2}, 2]$. For abs. min and abs. max,

first, we check the critical point(s), we have.

$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x} = 0 \Rightarrow x=1 \quad f(1) = 1 - \ln 1 = 1.$

$f'(x) = \frac{x-1}{x}$ DNE $\Rightarrow x=0$ (NOT in $[\frac{1}{2}, 2]$) $\times$

Then, checking $x = \frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{2} - \ln(\frac{1}{2}) = \frac{1}{2} + \ln 2 = 0.5 + 0.69 = 1.19.$

$x = 2$, $f(2) = 2 - \ln(2) = 2 - \ln 2 > 1$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \quad 2 - 0.69 = 1.31$

($\ln e = 1.$ $\underset{0.69}{\ln 2} < 1$, since $e = 2.71\cdots$)

$\Rightarrow$ abs. max is $f(2) = 1.31.$

abs. min is $f(1) = 1.$

§4.1

62. let f(x) be $e^{-x}-e^{-2x}$ on $[0,1]$.

Checking the critical number on $[0,1]$, we have

$$f'(x) = -e^{-x}+2e^{-2x} = -\frac{1}{e^x}+\frac{2}{e^{2x}} = \frac{-e^x+2}{e^{2x}} = 0 \Rightarrow e^x=2.$$

$$\Rightarrow x=\ln 2, \quad f(\ln 2) = e^{-\ln 2}-e^{-2\ln 2} = \frac{1}{2}-\frac{1}{4} = \frac{1}{4}.$$

Checking the endpoint $x=0$ $f(0)=1-1=0$

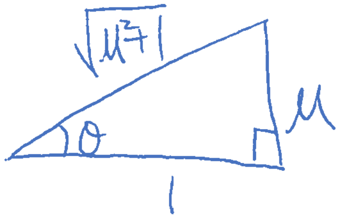
$x=1$, $f(1)=\frac{1}{e}-\frac{1}{e^2}=\frac{e-1}{e^2}=\frac{1.71}{7.38}<\frac{1}{4}$.

Thus, abs. max is $f(\ln 2)=\frac{1}{4}$

abs min is $f(0)=0$

70. Given $F=\dfrac{\mu W}{\mu\sin\theta+\cos\theta}$, $\theta\in[0,\frac{\pi}{2}]$.

We have $\dfrac{dF}{d\theta}=\dfrac{-(\mu\cos\theta-\sin\theta)\mu W}{(\mu\sin\theta+\cos\theta)^2}=0$



We obtain $\mu\cos\theta-\sin\theta=0 \Rightarrow \mu=\tan\theta$.

So, $F=\dfrac{\tan\theta\, W}{\tan\theta\cdot\sin\theta+\cos\theta}=\dfrac{\tan\theta\, W}{\frac{\sin^2\theta}{\cos\theta}+\cos\theta}=\dfrac{\cos\theta\tan\theta\, W}{\sin^2\theta+\cos^2\theta}=\sin\theta\, W.$

as $\mu=\tan\theta$

and if $\mu=\tan\theta$. $\sin\theta=\dfrac{\mu}{\sqrt{\mu^2+1}}$. we have $F=\dfrac{\mu W}{\sqrt{\mu^2+1}}$

For endpoints, we have $F(0)=\dfrac{\mu W}{1}=\mu W$

and $F(\frac{\pi}{2})=\dfrac{\mu W}{\mu}=W$

Since $\frac{u}{\sqrt{u^2+1}} \le 1$ and $\frac{u}{\sqrt{u^2+1}} \le u$, so $\frac{u}{\sqrt{u^2+1}}$ ϖ is the smallest value of the three values,

so, as $u = \tan\theta$, F has local & abs. min,

74. let $g(x) = 2 + (x-5)^3$.

We have $g'(x) = 3(x-5)^2 = 0 \Rightarrow x = 5$ is a critical number.

However, as $x > 5$. $g(x) > 2$, and as $x < 5$, $g(x) < 2$,

in the small interval around 5,

Thus, $g(x)$ has no local extreme value at 5.