Honors Calculus, Math 1450 – Assignment 2 Solution

(1) Let $r(t)$ be the radius of a sphere and $\frac{dr}{dt} = r^{\frac{1}{3}}$.

(a) Let $V(t)$ be the volume of the sphere. To find $\left.\frac{dV}{dt}\right|_{r=2}$,

We have $\boxed{V(t) = \frac{4}{3}\pi r(t)^3}$. Then, do "$\frac{d}{dt}$" on both sides, we obtain, by chain rule

$$\frac{dV(t)}{dt} = \frac{4}{3}\pi\, 3[r(t)]^2 \cdot \frac{dr(t)}{dt}$$

Thus $$\left.\frac{dV(t)}{dt}\right|_{r=2} = \frac{4}{3}\pi [2]^2 \cdot 3 \cdot \left. r^{\frac{1}{3}}\right|_{r=2} = 4\pi \cdot 4 \cdot 2^{\frac{1}{3}} = 16\sqrt[3]{2}\,\pi.$$

(b) Let $S(t)$ be the surface area of the sphere.

To find $\left.\frac{dS(t)}{dt}\right|_{r=2}$, we have $\boxed{S(t) = 4\pi (r(t))^2}$ and

$$\frac{dS(t)}{dt} = 4\cdot\pi\cdot 2\,(r(t))\cdot\frac{dr(t)}{dt}.$$ Then

$$\left.\frac{dS(t)}{dt}\right|_{r=2} = 4\pi \cdot 2 \cdot 2 \cdot \left. r^{\frac{1}{3}}\right|_{r=2} = 16\sqrt[3]{2}\,\pi$$

(2) $$\frac{d}{dx}[\csc(x)] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] \overset{\text{quotient rule}}{=} \frac{-1\cdot(\cos(x))}{\sin^2(x)} = -\frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)}$$

$$= -\cot(x)\cdot\csc(x).$$

(3) let $\sin(x)$ be the sine function with an angle x in degrees,
Let $\mathrm{SIN}(y)$ be the sine function with an angle y in radians
and we know $\lim\limits_{y\to 0} \frac{\mathrm{SIN}(y)}{y} = 1$.

Since $x \cdot \frac{\pi}{180} = y$. Then

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{y\to 0} \frac{\mathrm{SIN}(y)}{\frac{180}{\pi} y} = \lim_{y\to 0}\left(\frac{\mathrm{SIN}(y)}{y} \cdot \frac{\pi}{180}\right) = \frac{\pi}{180}.$$

if $x\to 0$, then $y \to 0$
and $x = \frac{180}{\pi} y$

(4) Find $\frac{dw}{dt}$ if

(a) $w = \tan(x)$, $x = 2t^2+1$, We have, by chain rule,

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} = \sec^2(x) \cdot 4t = \left[\sec^2(2t^2+1)\right] 4t.$$

(d) $w = 2^x$, $x = \sin(\sqrt{t})$, We have.

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} = \ln 2 \cdot 2^x \cdot \frac{1}{2} \frac{1}{\sqrt{t}} \cdot \cos(\sqrt{t})$$
$$= \ln 2 \; 2^{\sin(\sqrt{t})} \cdot \frac{\cos(\sqrt{t})}{2\sqrt{t}}$$

(5) Assume $f(x)$ is one-to-one and differentiable and its inverse f^{-1} is differentiable with $f'(x) \neq 0$ for any x.

To prove $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$. by def. of inverse function.

We have $x = f(f^{-1}(x))$. Then, do "$\frac{d}{dx}$" on both sides,

we obtain

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[f(f^{-1}(x))\right]$$

$$\Rightarrow \quad 1 = f'(f^{-1}(x)) \cdot (f^{-1})'(x) \quad \text{by chain rule.}$$

$$\Rightarrow \quad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \text{since } f'(x) \neq 0.$$

$\blacksquare$

(6) Since this is an "if and only if" proof, we need to consider both sides

($\Rightarrow$) Assume a is a double root of f.

To prove a is a root of $f'(x)$ and $f(x)$.

For $f(x)$, obviously, by assumption, a is a root of $f(x)$.

For $f'(x)$, let $f(x) = (x-a)^2 g(x)$ for some polynomial $g(x)$.

We have $f'(x) \underset{\substack{\uparrow \\ \text{product rule}}}{=} 2(x-a)g(x) + (x-a)^2 \cdot g'(x)$

$$= (x-a)\left[2g(x) + g'(x)\right]$$

which means a is a root of $f'(x)$.

(6)

($\Leftarrow$) Assume a is a root of both $f'(x)$ and $f(x)$.

To prove a is a double root of $f(x)$,

let $f(x) = (x-a)\,g(x)$, $f'(x) = (x-a)\,h(x)$, for some polynomials $g(x)$, $h(x)$.

If we can prove a is a root of $g(x)$, then we're done

Since $f(x) = (x-a)\,g(x)$, we have $f'(x) = g(x) + (x-a)\,g'(x)$.

By assumption, we have $f'(x) = (x-a)\,h(x)$, which implies

$$(x-a)\,h(x) = g(x) + (x-a)\,g'(x)$$

$$\Rightarrow \quad (x-a)\,h(x) - (x-a)\,g'(x) = g(x)$$

$$\Rightarrow \quad (x-a)\left(h(x) - g'(x)\right) = g(x)$$

Then a is a root of $g(x)$. $\blacksquare$

(6) Since $\lim\limits_{\theta\to 0} \dfrac{\sin(a\theta)}{a\theta} = 1$ for a constant a. Then

$$\lim_{\theta\to 0} \frac{\sin(7\theta)}{\sin(2\theta)} = \lim_{\theta\to 0} \frac{\sin(7\theta)}{7\theta}\cdot\frac{2\theta}{\sin(2\theta)}\cdot\frac{7}{2}$$

Since $\lim\limits_{\theta\to 0}\dfrac{\sin(7\theta)}{7\theta}$, $\lim\limits_{\theta\to 0}\dfrac{2\theta}{\sin(2\theta)}$ and $\lim\limits_{\theta\to 0}\dfrac{7}{2}$ exist

$$= \left[\lim_{\theta\to 0}\frac{\sin(7\theta)}{7\theta}\right]\cdot\left[\lim_{\theta\to 0}\frac{2\theta}{\sin(2\theta)}\right]\cdot\lim_{\theta\to 0}\frac{7}{2}$$

$$= 1\cdot 1\cdot \frac{7}{2} = \frac{7}{2}$$

(7) Given $f(x)$. Find $f'(x)$.

(i) $f(x)=\ln(\ln(x^2+1))$. By chain rule, we have

$$f'(x)=\frac{1}{\ln(x^2+1)}\cdot\frac{1}{x^2+1}\cdot 2x$$

(ii) $f(x)=e^{x^2\tan(x)}$. By chain rule, we have

$$f'(x)=(x^2\tan(x))'\cdot e^{x^2\tan(x)}=(2x\tan(x)+x^2\sec^2(x))\,e^{x^2\tan(x)}$$

(iii) $f(x)=x^{x^2}$. Take "ln" on both sides, we have

$$\ln f(x)=\ln x^{x^2}=x^2\ln x \quad \text{(by the property of ln. function)}$$

Do "$\frac{d}{dx}$" on both sides, we have

$$\frac{f'(x)}{f(x)}=2x\cdot\ln x+x^2\cdot\frac{1}{x}$$

$$\Rightarrow f'(x)=\left[2x\cdot\ln x+\frac{x^2}{x}\right]f(x)=\left[2x\cdot\ln x+x\right]x^{x^2}$$

(8) By def. of differentiable of f, we check the existence of the limit $\lim\limits_{h\to0}\frac{f(a+h)-f(a)}{h}$ $\left(\text{check } \lim\limits_{h\to0^+}\frac{f(a+h)-f(a)}{h}=\lim\limits_{h\to0^-}\frac{f(a+h)-f(a)}{h}\right)$

Assume $f(x)=x^{\frac{3}{2}}$.

① To prove $f(x)$ is differentiable at $x=0$. We have

$$\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^+}\frac{h^{\frac{3}{2}}-0}{h}=\lim_{h\to0^+}h^{\frac{1}{2}}=0 \quad \text{and}$$

(8) conti.

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{h^{\frac{3}{2}}-0}{h}=\lim_{h\to 0^-}h^{\frac{1}{2}}=0.$$

which imply $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ exists. Thus

f is differentiable at $x=0$.

② To prove $f'(x)$ is NOT differentiable at $x=0$, We have

(means $f(x)$ is NOT twice differentiable at $x=0$)

$f'(x)=\frac{3}{2}x^{\frac{1}{2}}$ and

$$\lim_{h\to 0^+}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0^+}\frac{\frac{3}{2}h^{\frac{1}{2}}-0}{h}=\lim_{h\to 0^+}\frac{3}{2}\frac{1}{h^{\frac{1}{2}}}\to\infty\ (\text{DNE})$$

so $f'(x)$ is NOT differentiable at $x=0$.

(9) Given $f(x)$. Find $f'(x)$.

(i) $f(x)=\ln(\ln(x^4+1))$, We have

$$f'(x)=\frac{1}{\ln(x^4+1)}\cdot\frac{1}{x^4+1}\cdot 4x^3$$

(ii) $f(x)=e^{x^2\sin(x)}$, We have

$$f'(x)=[x^2\sin(x)]'e^{x^2\sin(x)}=[2x\sin(x)+x^2\cos(x)]e^{x^2\sin(x)}.$$

(iii) $f(x)=\cot^2(x)$, We have

$$f'(x)=2\cot(x)\cdot[\cot(x)]'=-2\cot(x)\cdot\csc^2(x).$$

(10) Given $f(x)$, Find $f'(x)$.

(i) $f(x) = \ln(\sec(x) + \tan(x))$. We have

$$f'(x) = \frac{1}{\sec(x)+\tan(x)} \cdot [\sec(x)+\tan(x)]'$$

$$= \frac{\sec(x)\cdot\tan(x) + \sec^2(x)}{\sec(x)+\tan(x)} = \frac{\sec(x)[\tan(x)+\sec(x)]}{\sec(x)+\tan(x)}$$

$$= \sec(x)$$

(ii) $g(t) = e^{\frac{t}{t^2+6}}$. We have

$$g'(t) = \left(\frac{t}{t^2+6}\right)' e^{\frac{t}{t^2+6}} = \frac{t^2+6-2t\cdot t}{(t^2+6)^2} e^{\frac{t}{t^2+6}} = \frac{6-t^2}{(t^2+6)^2} e^{\frac{t}{t^2+6}}.$$

(11) Find $\frac{d^2y}{dx^2}$ for given equations:

(i) Given $x^3+y^3=1 \overset{\frac{d}{dx}}{\Longrightarrow} 3x^2+3y^2\frac{dy}{dx}=0$ $\left(\Rightarrow \frac{dy}{dx} = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}\right)$

$$\overset{\frac{d}{dx}}{\Longrightarrow} 6x + 6y\frac{dy}{dx}\cdot\frac{dy}{dx} + 3y^2\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow 6x + 6y\cdot\left(-\frac{x^2}{y^2}\right)\left(-\frac{x^2}{y^2}\right) + 3y^2\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow 6x + \frac{6x^4y}{y^4} + 3y^2\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{-6x+\frac{6x^4}{y^3}}{3y^3}$$

(11)

(ii) Given $y + \sin(y) = x$. $\overset{\frac{d}{dx}}{\Longrightarrow}$ $\frac{dy}{dx} + \cos(y)\frac{dy}{dx} = 1$ $\left(\Rightarrow \frac{dy}{dx} = \frac{1}{1+\cos(y)}\right)$

$\overset{\frac{d}{dx}}{\Longrightarrow}$ $\frac{d^2y}{dx^2} + (-\sin(y))\frac{dy}{dx}\cdot\frac{dy}{dx} + \cos(y)\frac{d^2y}{dx^2} = 0$

$\Rightarrow \frac{d^2y}{dx^2} - \sin(y)\cdot\left(\frac{1}{1+\cos(y)}\right)^2 + \cos(y)\frac{d^2y}{dx^2} = 0$

$\Rightarrow (1+\cos(y))\frac{d^2y}{dx^2} = \frac{\sin(y)}{(1+\cos(y))^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sin(y)}{(1+\cos(y))^3}$

(12) Given equation $x^2y - 5xy^2 = -6$. Find the tangent line at $(3,1)$.

Slope: $\left.\frac{dy}{dx}\right|_{(x,y)=(3,1)}$.

Do "$\frac{d}{dx}$" on the equation, we have

$2xy + x^2\frac{dy}{dx} - 5y^2 - 5x\cdot 2y\cdot\frac{dy}{dx} = 0$

$\Rightarrow (x^2 - 10xy)\frac{dy}{dx} = 5y^2 - 2xy \Rightarrow \frac{dy}{dx} = \frac{5y^2 - 2xy}{x^2 - 10xy}$

Then $\left.\frac{dy}{dx}\right|_{(x,y)=(3,1)} = \frac{5(1)^2 - 2\cdot 3\cdot 1}{3^2 - 10\cdot 3\cdot 1} = \frac{5-6}{9-30} = \frac{1}{21}$

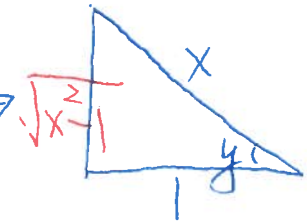
Thus, the tangent line is

$$y - 1 = \frac{1}{21}(x-3)$$

(13)

(a) Define $\sec^{-1}(x)$ by this

$y=\sec^{-1}(x) \Leftrightarrow \frac{x}{1}=\sec(y)$, $0 \le y < \frac{\pi}{2}$



We have $x=\sec(\sec^{-1}(x))$. Do "$\frac{d}{dx}$" on both sides, we obtain

$$1 = \sec(\sec^{-1}(x)) \tan(\sec^{-1}(x)) \cdot \frac{d}{dx}(\sec^{-1}(x))$$

$$\Rightarrow 1 = x \cdot \frac{\sqrt{x^2-1}}{1} \frac{d}{dx}(\sec^{-1}(x)) \Rightarrow \frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}.$$

(b) Define $\sec^{-1}(x)$ by this

$y=\sec^{-1}(x) \Leftrightarrow \sec(y)=x$, $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

Do "$\frac{d}{dx}$" on "$\sec(y)=x$" , we have

$$\sec(y)\tan(y)\frac{dy}{dx}=1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec(y)\tan(y)}$$

① As $0 \le y < \frac{\pi}{2}$, we have $\sec(y) = x (\ge 1)$
and $\tan(y) = \sqrt{x^2-1}$

② As $\frac{\pi}{2} < y \le \pi$, we have $\sec(y) = x (\le -1)$ and
$\tan(y) = -\sqrt{x^2-1}$

Combine ① and ② . We have $\frac{d}{dx}(\sec^{-1}(x)) = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x \ge 1; \\ \frac{1}{-x\sqrt{x^2-1}}, & x \le -1. \end{cases}$

$$\Rightarrow \frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{|x|\sqrt{x^2-1}} , |x| \ge 1$$

(14) Given $y=(2x+1)^3(x^4-2)^5$. Using logarithmic differentiable, to find y'

We have

$$\ln y = \ln\left[(2x+1)^3(x^4-2)^5\right]$$

$$\Rightarrow \ln y = \ln(2x+1)^3 + \ln(x^4-2)^5$$

$$\Rightarrow \ln y = 3\ln(2x+1) + 5\ln(x^4-2).$$

$$\overset{\frac{d}{dx}}{\Rightarrow} \frac{y'}{y} = \frac{3}{2x+1}\cdot 2 + \frac{5}{x^4-2}\cdot 4x^3$$

$$\Rightarrow \underline{y'} = \left[\frac{6}{2x+1} + \frac{20x^3}{x^4-2}\right]y = \underline{\left[\frac{6}{2x+1} + \frac{20x^3}{x^4-2}\right]\cdot(2x+1)^3(x^4-2)^5}$$

(15) Assume $f(x)=\ln(|x|)$, $x\neq 0$, to find $f'(x)$, we have

$$f(x)=\ln(|x|)=\begin{cases}\ln(x) & , x>0;\\ \ln(-x) & , x<0.\end{cases}$$

Then

$$f'(x)=\begin{cases}\frac{1}{x} & , x>0;\\ \frac{1}{-x}\cdot(-1), & x<0.\end{cases} \Rightarrow f'(x)=\begin{cases}\frac{1}{x} & , x>0;\\ \frac{1}{x} & , x<0.\end{cases}$$

Thus $f'(x)=\frac{1}{x}$ for all $x\neq 0$.