Honors Calculus, Math 1450 – Assignment I Solution

(1)

- $f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3}$
- $f(x) = x^{\pi} \Rightarrow f'(x) = \pi x^{\pi-1}$ (π is a constant)

(2) Finding $f'(-2)$ for $f(x)=x^3$ by definition of derivative,

We have $c=-2$ and

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0} \frac{(-2+h)^3-(-2)^3}{h}$$

$$= \lim_{h\to 0} \frac{-8+12h-6h^2+h^3-(-8)}{h} = \lim_{h\to 0} \frac{12h-6h^2+h^3}{h}$$

$$= \lim_{h\to 0} 12-6h+h^2 = 12$$

(3) Given $f(1)=1$, $g(1)=2$, $f'(1)=3$, $g'(1)=-3$. Find $(fg)'(1)$.

By Product Rule, we have $(fg)' = f'g + g'f$ and

$$(fg)'(1) = f'(1)\cdot g(1) + g'(1)\cdot f(1)$$

$$= 3\cdot 2 + (-3)\cdot(1) = 6-3 = 3$$

(4) Find $\frac{df}{dx}$ for given f.

(a) $f(x) = 2x^3$, $\quad \frac{df}{dx} = 6x^2$

(b) $f(x) = \frac{1}{x^2+1}$. By Quotient Rule, we have

$$\frac{df}{dx} = \frac{(1)' \cdot (x^2+1) - 1 \cdot (x^2+1)'}{(x^2+1)^2} = -\frac{2x}{(x^2+1)^2}$$

Another way: $f(x) = \frac{1}{x^2+1} = (x^2+1)^{-1}$. Then

By chain Rule, we have

$$f'(x) = -1 \cdot (x^2+1)^{-2} \cdot (x^2+1)' = -(x^2+1)^{-2} \cdot (2x) = -\frac{2x}{(x^2+1)^2}$$

(c) $f(x) = \frac{2x^3}{x+1}$. By Quotient Rule, we have

$$\frac{df}{dx} = \frac{(2x^3)'(x+1) - (2x^3)(x+1)'}{(x+1)^2} = \frac{6x^2(x+1) - 2x^3}{(x+1)^2} = \frac{4x^3+6x^2}{(x+1)^2}$$

(5) By Chain Rule,

$$\frac{d}{dx}\left[(f(x))^2+1\right] = 2(f(x)) \cdot f'(x).$$

Assume $|f(0)|<2$ and $|f'(0)|<1$, then

$$\left| \frac{d}{dx}\left[(f(x))^2+1\right]\Big|_{x=0} \right| = \left|2(f(0)) \cdot f'(0)\right| = 2|f(0)||f'(0)| < 2 \cdot 2 \cdot 1 = 4$$

(6) Suppose $g(x) = f(cx)$. By definition of derivative, we have

$$g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h \to 0} \frac{f(c(x+h))-f(cx)}{h}$$

$$= \lim_{h \to 0} \frac{f(cx+ch)-f(cx)}{h} = \lim_{h \to 0} c \cdot \frac{f(cx+ch)-f(cx)}{c \cdot h}$$

$$= c \cdot \lim_{h \to 0} \frac{f(cx+ch)-f(cx)}{ch} = c \cdot f'(cx).$$

(7) Find $f''(x)$.

(a) $f(x) = x^3 + 3x^2$. Then $f'(x) = 3x^2 + 6x \Rightarrow$

$f''(x) = 6x + 6$.

(b) Given $f'(x) = x^3$. Then $f''(x) = 3x^2$.

(c) Given $f(x) = ax^2 + bx + c$ and a, b, c are constants.

$f'(x) = 2ax + b \Rightarrow f''(x) = 2a$.

(8) Given curve $y=\frac{8}{x^2+x+2}$. Find tangent line of y at $x=2$.

For a line, we need the slope of the line and a point at the line.

① Slope at $x=2$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{x=2} \overset{\text{quotient rule}}{=} \left[\frac{(8)'(x^2+x+2)-(x^2+x+2)'\cdot 8}{(x^2+x+2)^2}\right]_{x=2}$

$$= \left.\frac{-8\cdot(2x+1)}{(x^2+x+2)^2}\right|_{x=2} = \frac{-8\cdot 5}{(2^2+2+2)^2} = \frac{-40}{(8)^2} = -\frac{5}{8}.$$

② Given point is $(2, y(2)) = \left(2, \frac{8}{2^2+2+2}\right) = (2,1)$

Then, by ①, ②, the equation of tangent line is

$$y-1 = -\frac{5}{8}(x-2).$$

(9) Suppose f is differentiable.

(a) $\lim\limits_{h\to 0}\frac{f(x+5h)-f(x)}{h} = \lim\limits_{h\to 0} 5\frac{f(x+5h)-f(x)}{5h} \overset{\substack{\text{if } h\to 0,\\ \text{then } 5h\to 0}}{=} \lim\limits_{5h\to 0} 5\frac{f(x+5h)-f(x)}{5h}$

$$= 5\lim_{5h\to 0}\frac{f(x+5h)-f(x)}{5h} = 5\cdot f'(x).$$

(b) $\lim\limits_{h\to 0}\frac{f(x)-f(x+h)}{h} = \lim\limits_{h\to 0} -\left[\frac{f(x+h)-f(x)}{h}\right] = -\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= -f'(x).$$

(10) Given curve $y=e^x$, Find the equation of normal line of y at $x=2$.

$$\boxed{\begin{array}{l}\text{Fact: Let } S_T \text{ be the slope of tangent line and} \\ \qquad\qquad S_n \text{ be the slope of normal line.} \\ \text{we have } S_T \cdot S_n = -1. \Rightarrow S_n = -\dfrac{1}{S_T}.\end{array}}$$

- Slope of normal line at $x=2$:

First, we find $S_T = \left.\frac{dy}{dx}\right|_{x=2} = \left.e^x\right|_{x=2} = e^2$.

Then $S_n = -\frac{1}{e^2}$.

- Point: $(2, y(2)) = (2, e^2)$.

Then the line is $y - e^2 = -\frac{1}{e^2}(x-2)$.

(11). Given $y = (x^{-1}+2x)^5$, Then by chain Rule.

$$\frac{dy}{dx} = 5(x^{-1}+2x)^4 \cdot (x^{-1}+2x)'$$
$$= 5(x^{-1}+2x)^4 \cdot \left[-x^{-2}+2\right]$$

(12) Given $xy+yx^2=x+y$. Then by Product rule, we have.

$$\frac{d}{dx}\left(xy+yx^2\right)=\frac{d}{dx}(x+y)$$

$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}(yx^2)=1+\frac{dy}{dX}$$

$$\Rightarrow y+x\frac{dy}{dx}+x^2\frac{dy}{dx}+2Xy=1+\frac{dy}{dx}.$$

$$\Rightarrow y+2xy-1=\frac{dy}{dx}-x\frac{dy}{dx}-x^2\frac{dy}{dx}.$$

$$\Rightarrow y+2xy-1=(1-x-x^2)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y+2xy-1}{1-x-x^2}$$

Thus, $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=\left.\frac{y+2xy-1}{1-x-x^2}\right|_{(x,y)=(1,1)}=\frac{1+2\cdot1\cdot1-1}{1-1-1^2}$

$$=\frac{2}{-1}=-2.$$

(13). Given a curve $y=\frac{1-x}{x+1}$ and a line $3x+2y=1$.
To Find a tangent line of y such that this line is parallel to $3x+2y=1$, it means these two lines have the same slope.

$3x+2y=1 \Rightarrow 2y=1-3x \Rightarrow y=-\frac{3}{2}x+\frac{1}{2} \Rightarrow$
the slope of this line is $-\frac{3}{2}$.

Then, find x such that $y'(x)=-\frac{3}{2}$, we have

$$y'(x)=\frac{(1-x)'(x+1)-(1-x)(x+1)'}{(x+1)^2}=\frac{-(x+1)-(1-x)}{(x+1)^2}$$

$$=\frac{-x-1-1+x}{(x+1)^2}=-\frac{2}{(x+1)^2}$$

Thus $-\frac{2}{(x+1)^2}=-\frac{3}{2} \Rightarrow 4=3(x+1)^2 \Rightarrow (x+1)^2=\frac{4}{3}$.

$$\Rightarrow x+1=\pm\frac{2}{\sqrt{3}} \Rightarrow x=-1\pm\frac{2}{\sqrt{3}}$$

- As $x=-1+\frac{2}{\sqrt{3}}$, we have $y=\frac{2-\frac{2}{\sqrt{3}}}{\frac{2}{\sqrt{3}}}=\frac{\sqrt{3}}{2}\left(2-\frac{2}{\sqrt{3}}\right)=\sqrt{3}-1$. and tangent line is $\underline{y-(\sqrt{3}-1)=-\frac{3}{2}\left[x-\left(-1+\frac{2}{\sqrt{3}}\right)\right]}$
- As $x=-1-\frac{2}{\sqrt{3}}$, we have $y=\frac{2+\frac{2}{\sqrt{3}}}{-\frac{2}{\sqrt{3}}}=-\frac{\sqrt{3}}{2}\left(2+\frac{2}{\sqrt{3}}\right)=-\sqrt{3}-1$ and tangent line is $\underline{y-(-\sqrt{3}-1)=-\frac{3}{2}\left[x-\left(-1-\frac{2}{\sqrt{3}}\right)\right]}$.

(14). Let $y = x^2 \cdot \tan^{-1}(2x)$. By Chain Rule and Product Rule,

$$y' = (x^2)' \tan^{-1}(2x) + x^2 [\tan^{-1}(2x)]'$$

$$= 2x \cdot \tan^{-1}(2x) + x^2 \frac{1}{1+(2x)^2} \cdot (2x)'$$

$$= 2x \cdot \tan^{-1}(2x) + \frac{2x^2}{1+(2x)^2}$$

(15). Let $y = \tan^2(\sin\theta)$, We have $(y = [\tan(\sin\theta)]^2$

$$y' = 2[\tan(\sin\theta)] \cdot (\tan(\sin\theta))'$$

$$= 2 \cdot [\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot [\sin\theta]'$$

$$= 2 \cdot [\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta.$$

(16). Let $y = \cos^2(4x) + \sin^2(2x)$. We obtain

$$\frac{dy}{dx} = 2[\cos(4x)] \cdot [\cos(4x)]' + 2[\sin(2x)] \cdot [\sin(2x)]'$$

$$= 2\cos(4x) \cdot [-4\sin(4x)] + 2\sin(2x) \cdot 2\cos(2x)$$

$$= -8\cos(4x)\sin(4x) + 4\sin(2x)\cos(2x).$$

Find the tangent line at $x = \frac{\pi}{4}$, we have

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -8\cos(4 \cdot \tfrac{\pi}{4}) \cdot \sin(4 \cdot \tfrac{\pi}{4}) + 4\sin(2 \cdot \tfrac{\pi}{4})\cos(2 \cdot \tfrac{\pi}{4}) = 0$$

and point $(\frac{\pi}{4}, y(\frac{\pi}{4})) = (\frac{\pi}{4}, 2)$, then tangent line is $y = 2$.

(17) If $f(x) = e^{\sin(x)}$, we have

$$f'(x) = (\sin(x))' \cdot e^{\sin(x)} = \cos(x) \cdot e^{\sin(x)}.$$

Since $e^{\sin(x)} > 0 \ \forall x \in \mathbb{R}$ and $-1 \leq \cos(x) \leq 1 \ \forall x \in \mathbb{R}$.

We have $\cos(x) \cdot e^{\sin(x)} \leq e^{\sin(x)}$

$$\Rightarrow \quad f'(x) \leq f(x).$$

(18). The function of position of x w.r.t time is

$$X(t) = \sqrt{1+4t^2}, \quad \text{for } t \geq 0. \quad \left(X(t) = (1+4t^2)^{\frac{1}{2}}\right)$$

Then velocity is $X'(t) = \frac{dx}{dt} = \frac{1}{2}(1+4t^2)^{-\frac{1}{2}} \cdot 8t$

$$= \frac{1}{2} \frac{1}{\sqrt{1+4t^2}} \cdot 8t = \frac{4t}{\sqrt{1+4t^2}}$$

and acceleration is $X''(t) = \frac{d^2x}{dt^2} = \left[\frac{1}{2}(1+4t^2)^{-\frac{1}{2}} \cdot 8t\right]'$

product Rule

$$= 4 \cdot (1+4t^2)^{-\frac{1}{2}} + 4t \cdot \left(-\frac{1}{2}\right)(1+4t^2)^{-\frac{3}{2}} \cdot (8t)$$

$$= 4 \cdot (1+4t^2)^{-\frac{1}{2}} - 16t^2(1+4t^2)^{-\frac{3}{2}}.$$

To find limit velocity, we have

$$\lim_{t\to\infty} \frac{4t}{\sqrt{1+4t^2}} = \lim_{t\to\infty} \frac{\frac{1}{4t} \cdot 4t}{\frac{1}{4t}\sqrt{1+4t^2}} = \lim_{t\to\infty} \frac{1}{\sqrt{\frac{1}{16t^2} + \frac{1}{4}}} = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{1}{2}} = 2$$

(19) Given the trajectory of a particle $\frac{y(t)^2}{4} + x(t)^2 = 1$.

To find the point(s) at which the velocity in the vertical direction equals the velocity in the horizontal direction, it is sufficient to have a point (x, y) such that

$$\frac{dx}{dt} = \frac{dy}{dt}, \quad \text{or} \quad \frac{dy}{dx} = 1.$$

Then, do $\frac{d}{dt}$ on $\frac{y^2}{4} + x^2 = 1$, we have

$$\frac{1}{4} \cdot 2y \cdot \frac{dy}{dt} + 2x \frac{dx}{dt} = 0 \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2x}{\frac{y}{2}} = \frac{-4x}{y} = 1.$$

$\Rightarrow -4x = y$, use this equality we have

$$\frac{(-4x)^2}{4} + x^2 = 1 \Rightarrow 4x^2 + x^2 = 1 \Rightarrow 5x^2 = 1. \Rightarrow x = \pm\sqrt{\frac{1}{5}}.$$

There, $x = \frac{1}{\sqrt{5}}, y = \frac{-4}{\sqrt{5}}$ or $x = -\frac{1}{\sqrt{5}}, y = \frac{4}{\sqrt{5}}$,

(20) Given the differential equation:

$$\frac{d^2y}{dt^2} = -y. \text{ --- } (*)$$

To check (a) $y=\sin(t)$ is a solution of $(*)$. we have

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\cos(t)\right) = -\sin(t) \text{ and}$$

$-y = -\sin(t)$ which is equal to $\frac{d^2y}{dt^2}$.

So $y=\sin(t)$ is a sol. of $(*)$.

To check (b) $y=\cos(t)$ is a solution of $(*)$, we have

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(-\sin(t)\right) = -\cos(t) \text{ and}$$

$-y = -\cos(t)$ which is exact $\frac{d^2y}{dt^2}$.

Find the solution of $\frac{d^2y}{dt^2} = -4y$. since this differetial equation has similar form of $(*)$. we can guess the solutions are $y=\sin(at)$ and $y=\cos(at)$ for an undetermined constant a.

For $y=\sin(at)$, we have $\frac{d^2y}{dt^2} = -a^2\sin(at)$ and $-4y = -4\sin(at)$

$$\Rightarrow -a^2\sin(at) = -4\sin(at) \Rightarrow a^2=4 \Rightarrow a=\pm 2.$$

So $y=\sin(2t)$ and $y=\sin(-2t)$ are the solutions, Similarly,

$y = \cos(2t)$ and $y = \cos(-2t)$ are the solutions.