Math 1431, Calculus I Final Review, Spring 2015.

Shang-Huan Chiu

May 4, 2015

1 Existence of Limit

Given a real function $f(x)$. Then $\lim_{x\to a} f(x)$, the limit of $f(x)$ at $x = a$, exists if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x),$$

so we have

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

2 Properties of Limit

2.1 Basic Properties

Given real functions $f(x)$, $g(x)$. Assume the limits of f, g exist at a point a, i.e. $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist.

- $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
- $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
- $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)]$.
- If $\lim_{x\to a} g(x) \neq 0$, we have $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.
- If f is a continuous function on $\mathbb{R}$, then we have $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$.

  Example 2.1 Let $f(x) = e^x$. Then $\lim_{x\to a} e^{g(x)} = e^{\lim_{x\to a} g(x)}$.

2.2 Special Property

Given two polynomials $P(x)$, $Q(x)$. Define $\deg(P)$ be the degree of polynomial $P(x)$ which is the highest degree of its terms. Then we have

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)} = \begin{cases} 0, & \text{if } \deg(P(x)) < \deg(Q(x)), \\ \text{Leading Coefficient}, & \text{if } \deg(P(x)) = \deg(Q(x)), \\ \infty, & \text{if } \deg(P(x)) > \deg(Q(x)). \end{cases}$$

where the leading coefficient is the ratio of the cofficients of the highest degree terms of $P(x)$ and $Q(x)$.

3 Continuity and Discontinuity of A Real Function

Definition 3.1 Given a real function $f(x)$. Then f is said to be continuous at a point a if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x-a| < \delta \text{ implies } |f(x) - f(a)| < \epsilon.$$

3.1 Classification

To classify the continuity or discontinuity of a real function at a point a, we can check the three values

$$(1) \lim_{x\to a^+} f(x),\ (2) \lim_{x\to a^-} f(x),\ (3) f(a)$$

and we have the following four conditions:

- If (1), (2), (3) exist and (1) = (2) = (3), then f is **continuous** at a.
- If (1), (2) exist and (1) = (2) $\neq$ (3), then f has a **removable discontinuity** at a.
- If (1), (2) exist and (1) $\neq$ (2), then f has a **jump discontinuity** at a.
- If at least one of the first two values tends to ∞ or $-\infty$, then f has an **infinite discontinuity** at a.

4 The Intermediate Value Theorem

Theorem 4.1 *If $f(x)$ is continuous on the closed interval $[a, b]$ and N is a value between $f(a)$ and $f(b)$, then there is at least one value c in (a, b) such that*

$$f(c) = N.$$

Corollary 4.2 (Root finding) *If $f(x)$ is continuous on the closed interval $[a, b]$ and $f(a)$, $f(b)$ have different signs, that is*

$$f(a) < 0 < f(b), \text{ or } f(b) < 0 < f(a),$$

then there is at least one value c in (a, b) such that

$$f(c) = 0.$$

5 The Pinching Theorem

Theorem 5.1 *Suppose $f(x), g(x)$, and $h(x)$ are defined in an open interval containing $x = a$. If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then*

$$\lim_{x\to a} g(x) = L.$$

5.1 Application of Pinching Theorem

- $\lim_{x\to 0} \frac{sin(ax)}{ax} = 1.$
- $\lim_{x\to 0} \frac{1-\cos(ax)}{ax} = 0.$

(We also can use L'Hôpital's rule to find these.)

6 Sum and Difference Formulas

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B).$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B).$$

7 Derivative

7.1 Definition of derivative

A function $f(x)$ is differentiable at x if and only if

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. We denote

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

and we refer to $f'(x)$ as the derivative of f at x.

7.2 Product Rule

Assume $f(x), g(x)$ are differentiable. Then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

7.3 Quotient Rule

Assume $f(x), g(x)$ are differentiable. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g(x)'}{[g(x)]^2}.$$

(lo D-hi minus hi D-lo)

7.4 Chain Rule

Assume $f(x), g(x)$ are differentiable. Then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Application of Chain Rule: Assume $u(x)$ is differentiable.

Function	Derivative of given function
$[u(x)]^n$, $n \neq 0$	$n \cdot u'(x)[u(x)]^{n-1}$
$\ln u(x)$	$\frac{u'(x)}{u(x)}$
$e^{u(x)}$	$u'(x)e^{u(x)}$
$\sin(u(x))$	$u'(x)\cos(u(x))$
$\cos(u(x))$	$-u'(x)\sin(u(x))$
$\tan(u(x))$	$u'(x)\sec^2(u(x))$
$\sec(u(x))$	$u'(x)\sec(u(x))\tan(u(x))$
$\cot(u(x))$	$-u'(x)\csc^2(u(x))$
$\csc(u(x))$	$-u'(x)\csc(u(x))\cot(u(x))$
$\sinh(u(x))$	$u'(x)\cosh(u(x))$
$\cosh(u(x))$	$u'(x)\sinh(u(x))$
$\arcsin(u(x))$	$\frac{u'(x)}{\sqrt{1-[u(x)]^2}}$
$\arctan(u(x))$	$\frac{u'(x)}{1+[u(x)]^2}$
$\text{arcsec}(u(x))$	$\frac{u'(x)}{\lvert u(x)\rvert\sqrt{[u(x)]^2-1}}$
$a^{u(x)}$,$a > 0$	$\ln(a)u'(x)a^{u(x)}$

Figure 1: Differentiation Table

(Please check the test4 review sheet table1)

8 Special Formulas for Areas and Volumes

- Area of a **circle** with a radius r:
$$A = \pi r^2.$$
- Volume of a **cone** with height h and radius of base r:
$$V = \frac{1}{3}\pi r^2 h.$$
- Volume of a **ball** with radius r:
$$V = \frac{4}{3}\pi r^3.$$
- Area of **sphere** with radius r:
$$S = 4\pi r^2.$$
- Surface area of **cylinder** with height h and radius of base r:
$$S = 2\pi rh + 2\pi r^2.$$

9 Mean Value Theorem

Theorem 9.1 (Mean Value Theorem) *If f is continuous on a $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 9.2 (Rolle's Theorem) *Let f be continuous on a $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there is at least one number c in (a, b) such that*

$$f'(c) = 0$$