Math 1431 Exam 2 Review

1. a. $\lim_{x\to 0}\frac{\sin 4x}{5x} = \lim_{x\to 0}\frac{\sin 4x}{5x}\cdot\frac{4}{4} = \frac{4}{5}$

b. $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0}\frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)} = \lim_{x\to 0}\frac{x}{x(\sqrt{4+x}+2)} = \frac{1}{4}$

c. $\lim_{x\to 0}\frac{\left(\frac{1}{x+1}-1\right)}{x} \overset{(\frac{0}{0})}{=} \lim_{x\to 0}\frac{\left(\frac{1}{x+1}-1\right)\cdot(x+1)}{x\cdot(x+1)} \overset{(\frac{0}{0})}{=} \lim_{x\to 0}\frac{1-(x+1)}{x(x+1)} \overset{(\frac{0}{0})}{=} \lim_{x\to 0}\frac{-x}{x(x+1)} = -1$

d. $\lim_{x\to -3}\frac{x^2+x-6}{x^2-9} \overset{(\frac{0}{0})}{=} \lim_{x\to -3}\frac{(x+3)(x-2)}{(x+3)(x-3)} = \lim_{x\to -3}\frac{(x-2)}{(x-3)} = \frac{5}{6}$

e. $\lim_{x\to 0} x\left(2-\frac{1}{x}\right) = \lim_{x\to 0} x\cdot\frac{2x-1}{x} = \lim_{x\to 0} 2x-1 = -1$

f. $\lim_{x\to 0}\frac{2\sin(x)\cos(x)}{2x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cos(x) = 1\cdot 1 = 1$

g. $\lim_{x\to 0}\frac{5x}{\tan(9x)} \overset{(\frac{0}{0})}{=} \lim_{x\to 0}\frac{5x}{\sin(9x)}\cdot\cos(9x) = \lim_{x\to 0}\frac{5x}{\sin 9x}\cdot\frac{9}{9}\cdot\cos(9x) = \frac{5}{9}$

h. $\lim_{x\to 0}\frac{\sin(x^2)}{6x} = \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot\frac{x^2}{6x} = \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot\frac{x}{6} = 1\cdot 0 = 0$

2. a. $f(x)$ is continuous on $(-\infty,1)\cup(1,\infty)$, we only need to check $f(x)$ at $x=1$: ① $\lim_{x\to 1^+} f(x)$, ② $\lim_{x\to 1^-} f(x)$, and ③ $f(1)$.

① $\lim_{x\to 1^+} f(x) = 1^3 = 1$, ② $\lim_{x\to 1^-} f(x) = 1^2+1 = 2$ ③ $f(1) = 8$

① ② exist and ① ≠ ② ⇒ Jump discontinuity at $x=1$.

2. b. f is continuous on $(-\infty,2)\cup(2,\infty)$. We only need to check things at $x=2$: ① $\lim_{x\to2^+} f(x)$, ② $\lim_{x\to2^-} f(x)$, and ③ $f(2)$

$$\lim_{x\to2^+} f(x) = 2^3=8 \qquad \lim_{x\to2^-} f(x) = 2\cdot2^2=8 \qquad f(2)=8$$

①②③ exist and ①=②=③ $\Rightarrow$ Continuous at $x=2$.

c. f is continuous on $(-\infty,-2)\cup(-2,\infty)$, we only need to check things at $x=-2$: ① $\lim_{x\to-2^+} f(x)$, ② $\lim_{x\to-2^-} f(x)$, and ③ $f(-2)$

$$\lim_{x\to-2^+} f(x) = (-2)^2-5=-1 \qquad \lim_{x\to-2^-} f(x) = 5-(-2)=7 \qquad f(-2)=7.$$

①② exist but ①≠② $\Rightarrow$ Jump discontinuity.

d. f is continuous on $(-\infty,-1)\cup(-1,\infty)$, we only need to check things at $x=-1$ ① $\lim_{x\to-1^+} f(x)$, ② $\lim_{x\to-1^-} f(x)$, and ③ $f(-1)$.

$$\lim_{x\to-1^+} f(x) = -1+2=1 \qquad \lim_{x\to-1^-} f(x) = (-1)^2=1 \qquad f(-1) = -1+2=1.$$

①②③ exist, ①=②=③ $\Rightarrow$ continuous at $x=-1$.

3. Check ① $\lim_{x\to3^+} f(x)$, ② $\lim_{x\to3^-} f(x)$, and ③ $f(3)$

$$\lim_{x\to3^+} f(x) = 3+1=4 \qquad \lim_{x\to3^-} f(x) = 3+1=4 \qquad f(3)=K$$

Continuous at $x=3$ $\Leftrightarrow$ ①=②=③ $\Rightarrow K=4$

4. Check (1) $\lim_{x\to -1^+} f(x)$, (2) $\lim_{x\to -1^-} f(x)$, and (3) $f(-1)$

$$\lim_{x\to -1^+} f(x) = -1\cdot B+3 \qquad \lim_{x\to -1^-} f(x) = 6\cdot(-1)^2-1=5 \qquad f(-1) = A.$$

Continuous at $x=-1 \iff (1)=(2)=(3) \iff \begin{cases} -B+3=5 \\ A=5 \end{cases} \Rightarrow \begin{cases} A=5 \\ B=-2 \end{cases}$

5. To check the existence of $f'(3)$. (differentiablity implies continuity)

First, check $f(x)$ is continuous at $x=3$.

(1) $\lim_{x\to 3^+} f(x) = 2\cdot 3=6$ (2) $\lim_{x\to 3^-} f(x) = 3^2-3=6$ (3) $f(3)=3^2-3=6$

$\Rightarrow (1)=(2)=(3)$ & exist $\Rightarrow$ continuous at $x=3$.

Then, check $f(x)$ is differentiable at $x=3$.

(a) $$\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+}\frac{2(3+h)-6}{h} = \lim_{h\to 0^+}\frac{2h}{h} = 2$$

(b) $$\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-}\frac{(3+h)^2-3-6}{h} = \lim_{h\to 0^-}\frac{6h+h^2}{h} = 6$$

Since (a) $\neq$ (b) $\Rightarrow f'(3)$ doesn't exist.

6.

a. Given $f(x)=3(2x-1)^4$. Then

$$f'(x) \underset{\text{chain rule}}{=} 3\cdot 4(2x-1)^3\cdot[2x-1]' = 3\cdot 4(2x-1)^3\cdot 2$$
$$= 24(2x-1)^3$$

b. Given $f(x)=\sec^3(2x)$. Then

$$f'(x) \underset{\text{chain rule}}{=} 3(\sec(2x))^2\cdot[\sec(2x)]' = 3\cdot\sec^2(2x)\cdot[\sec(2x)\tan(2x)]\cdot 2$$
$$= 6\cdot\tan(2x)\cdot\sec^3(2x).$$

6. c. $f(x) = 3\sqrt{x} + \frac{5}{x} = 3 \cdot x^{\frac{1}{2}} + 5 \cdot x^{-1}$.

$\Rightarrow f'(x) = \frac{3}{2} x^{-\frac{1}{2}} - 5 \cdot x^{-2} = \frac{3}{2} \frac{1}{\sqrt{x}} - \frac{5}{x^2}$.

d. $f(x) = \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

e. $f(x) = \frac{\sqrt{x} + 2x}{x^2} = \frac{\sqrt{x}}{x^2} + \frac{2x}{x^2} = \frac{x^{\frac{1}{2}}}{x^2} + 2 \cdot x^{-1} = x^{-\frac{3}{2}} + 2 \cdot x^{-1}$.

$\Rightarrow f'(x) = -\frac{3}{2} x^{-\frac{5}{2}} - 2x^{-2} = -\frac{3}{2} \cdot \frac{1}{x^{\frac{5}{2}}} - 2 \frac{1}{x^2}$.

f. $f(x) = (x^2 + 2x)^4 (x-1)^3$. Then

$f'(x) \underset{\substack{\uparrow \\ \text{product rule}}}{=} [(x^2+2x)^4]'(x-1)^3 + (x^2+2x)^4 [(x-1)^3]'$

$\underset{\substack{\uparrow \\ \text{chain rule}}}{=} 4(x^2+2x)^3 (2x+2)(x-1)^3 + (x^2+2x)^4 3(x-1)^2 \cdot 1$

g. $f(x) = \frac{(x-1)(x+1)}{x+2} = \frac{x^2-1}{x+2}$ we can use quotient rule or simplify f first:

$$\begin{array}{rrr|l} 1 & +0 & -1 & \underline{-2} \\ + & -2 & +4 & \\ \hline 1 & -2 & \boxed{3} & \end{array} \Rightarrow (x^2-1) = (x+2)(x-2) + 3 \Rightarrow f(x) = \frac{x^2-1}{x+2} = \frac{(x+2)(x-2)+3}{x+2}$$

$= \frac{(x-2)(x+2)}{(x+2)} + \frac{3}{(x+2)}$

$= x - 2 + 3(x+2)^{-1}$.

$\Rightarrow f'(x) = 1 - 3 \cdot (x+2)^{-2} = 1 - \frac{3}{(x+2)^2}$

h. $y = x \cdot \sqrt{x^3 + 5x}$

$y' \underset{\substack{\uparrow \\ \text{product rule}}}{=} \sqrt{x^3+5x} + x\left[\sqrt{x^3+5x}\right]' = \sqrt{x^3+5x} + x \cdot \frac{1}{2} \cdot (x^3+5x)^{-\frac{1}{2}} \cdot (3x^2+5)$

$= \frac{2(x^3+5x) + x(3x^2+5)}{2\sqrt{x^3+5x}} = \frac{5x^3+15x}{2\sqrt{x^3+5x}}$

6. i. $f(x)=\dfrac{1+\cos(x)}{1-\cos(x)}$. Then, by quotient rule, we have

$$f'(x)=\frac{[1+\cos(x)]'(1-\cos(x))-[1-\cos(x)]'(1+\cos(x))}{(1-\cos(x))^2}$$

$$=\frac{-\sin(x)(1-\cos(x))-\sin(x)(1+\cos(x))}{(1-\cos(x))^2}$$

$$=\frac{-\sin(x)+\sin(x)\cos(x)-\sin(x)-\sin(x)\cos(x)}{(1-\cos(x))^2}=\frac{-2\sin(x)}{(1-\cos(x))^2}$$

j. $f(x)=\sin^4(4x^2-6x+1)=[\sin(4x^2-6x+1)]^4$. Then, by chain rule,

$$f'(x)=4[\sin(4x^2-6x+1)]^3\cdot[\cos(4x^2-6x+1)]\cdot(8x-6)$$

k. $y=\dfrac{\cot(x)}{x^2}$ ~~= cot(x)~~ Then, by quotient rule, we have

$$y'=\frac{[\cot(x)]'x^2-(x^2)'\cot(x)}{[x^2]^2}=\cancel{\cot(x)}$$

$$=\frac{[-\csc^2(x)]x^2-2x\cot(x)}{x^4}=\frac{-x^2\csc^2(x)-2x\cot(x)}{x^4}$$

l. $f(\theta)=\sec(\theta)-\tan(\theta)$. Then

$$f'(\theta)=\sec(\theta)\tan(\theta)-\sec^2(\theta)$$

7. Find $\frac{dy}{dx}$ by implicit differentiation.

a. $x^2+y^2-4x+3y=7$, Then

$$\frac{d}{dx}(x^2+y^2-4x+3y)=\frac{d}{dx}(7)$$

$$\Rightarrow 2x+2y\cdot\frac{dy}{dx}-4+3\cdot\frac{dy}{dx}=0 \Rightarrow \underbrace{(2x-4)+(2y+3)\frac{dy}{dx}}_{\text{Collect the terms which have "}\frac{dy}{dx}\text{"}}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(2x-4)}{2y+3}=\frac{-2x+4}{2y+3}$$

b. $\sin(x)-\cos(y)-2=0$, Then

$$\frac{d}{dx}\left(\sin(x)-\cos(y)-2\right)=0 \Rightarrow \cos(x)-(-\sin(y))\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\cos(x)}{\sin(y)}$$

c. $x^3-xy+y^3=1$. Then

$$\frac{d}{dx}(x^3-xy+y^3)=\frac{d}{dx}(1) \Rightarrow 3x^2\underbrace{-y-x\frac{dy}{dx}}_{\text{chain rule of }(xy)'\text{ w.r.t }[x]}+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow (3x^2-y)+(-x+3y^2)\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-(3x^2-y)}{-x+3y^2}$$

d. $y\sqrt{x}-x\sqrt{y}=16$, Then

$$\frac{d}{dx}(y\cdot x^{\frac{1}{2}}-xy^{\frac{1}{2}})=\frac{d}{dx}(16) \Rightarrow \frac{dy}{dx}x^{\frac{1}{2}}+y\cdot\frac{1}{2}x^{-\frac{1}{2}}-y^{\frac{1}{2}}-x\cdot\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0$$

$$\Rightarrow \left(x^{\frac{1}{2}}-\frac{x}{2}\cdot\frac{1}{\sqrt{y}}\right)\frac{dy}{dx}+\left(\frac{y}{2\sqrt{x}}-\sqrt{y}\right)=0 \Rightarrow \frac{dy}{dx}=\frac{-\frac{y}{2\sqrt{x}}+\sqrt{y}}{\sqrt{x}-\frac{x}{2\sqrt{y}}}$$

1.

e. $xy=10$, Then

$$\frac{d}{dx}(xy)=\frac{d}{dx}(10) \Rightarrow y+x\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x}$$

f. $x\sin(2y)=1$, Then

$$\frac{d}{dx}(x\sin(2y))=\frac{d}{dx}(1) \Rightarrow \sin(2y)+x\cos(2y)\cdot 2\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\sin(2y)}{2x\cos(2y)}$$

g. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=5$, Then

$$\frac{d}{dx}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=\frac{d}{dx}(5) \Rightarrow \frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}}=\frac{-x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}=\frac{-\sqrt[3]{y}}{\sqrt[3]{x}}$$

h. $\cos(x+y)=4xy \Rightarrow \cos(x+y)-4xy=0$, Then,

$$\frac{d}{dx}\left(\cos(x+y)-4xy\right)=\frac{d}{dx}(0) \Rightarrow -\sin(x+y)\cdot\frac{d}{dx}(x+y)-4y-4x\frac{dy}{dx}=0$$

$$\Rightarrow -\sin(x+y)\cdot\left[1+\frac{dy}{dx}\right]-4y-4x\frac{dy}{dx}=0$$

$$\Rightarrow -\sin(x+y)-\sin(x+y)\frac{dy}{dx}-4y-4x\frac{dy}{dx}=0$$

$$\Rightarrow (-4x-\sin(x+y))\frac{dy}{dx}+(-4y-\sin(x+y))=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{4y+\sin(x+y)}{-4x-\sin(x+y)}$$

8. Use the definition of derivative to find derivative.

a. Given $f(x) = 3x^2 - x + 2$,

$3(x+h)^2 = 3(x^2+2xh+h^2)$

Then $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{3(x+h)^2 - (x+h) + 2 - (3x^2 - x + 2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-x} - h \cancel{+2} \cancel{-3x^2} \cancel{+x} \cancel{-2}}{h} = \lim\limits_{h\to 0} \dfrac{6xh + 3h^2 - h}{h}$

$= \lim\limits_{h\to 0} 6x + 3h - 1 = 6x - 1$.

b. Given $f(x) = \dfrac{2}{(x+5)}$.

Then $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{2}{(x+h+5)} - \dfrac{2}{(x+5)}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\dfrac{2[(x+5) - (x+h+5)]}{(x+5)(x+h+5)}}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{-2h}{(x+5)(x+h+5)}}{h}$

$= \lim\limits_{h\to 0} \dfrac{-2}{(x+5)(x+h+5)} = \dfrac{-2}{(x+5)^2}$

c. Given $f(x) = \sqrt{x+1}$

Then $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \dfrac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$

$= \lim\limits_{h\to 0} \dfrac{(x+h+1) - (x+1)}{h\cdot(\sqrt{x+h+1} + \sqrt{x+1})} = \lim\limits_{h\to 0} \dfrac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$

$= \dfrac{1}{2\sqrt{x+1}}$

9. $\frac{d^3}{dx^3}\left(\frac{3}{4}x^4-2x^3+x-10\right)=\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{3}{4}x^4-2x^3+x-10\right)\right)\right)$

$=\frac{d}{dx}\left(\frac{d}{dx}\left(3x^3-6x^2+1\right)\right)=\frac{d}{dx}\left(9x^2-12x\right)=18x-12$

10. Find $\frac{dy}{dx}$ at $x=-2$ for $y=(4x+1)(1-x)^3$

$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}\left((4x+1)(1-x)^3\right)\overset{\text{product rule}}{=}4(1-x)^3+(4x+1)\cdot 3(1-x)^2\cdot(-1)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=-2}=4(1+2)^3+(-8+1)\cdot 3(1+2)^2\cdot(-1)$

$=4\cdot 27+7\cdot 27=297.$

11. Find $\frac{d^2y}{dx^2}$ at point $(-2,1)$ for $x^2-y^2=3$.

First, to find $\frac{dy}{dx}$, we have

$\frac{d}{dx}(x^2-y^2)=\frac{d}{dx}(3) \Rightarrow \underbrace{2x-2y\cdot\frac{dy}{dx}=0}_{(*)} \Rightarrow \frac{dy}{dx}=\frac{x}{y}. \quad (**)$

Using $(*)$ to find $\frac{d^2y}{dx^2}$,

$\frac{d}{dx}\left(2x-2y\cdot\frac{dy}{dx}\right)=\frac{d}{dx}(0) \Rightarrow 2-\left(2\frac{dy}{dx}\frac{dy}{dx}+2y\frac{d^2y}{dx^2}\right)=0$

by $(**)$, since $\frac{dy}{dx}=\frac{x}{y}$, we have

$2-2\frac{x}{y}\cdot\frac{x}{y}-2y\frac{d^2y}{dx^2}=0 \Rightarrow \frac{d^2y}{dx^2}=\frac{2-\frac{2x^2}{y^2}}{2y}$

at $(-2,1)$, we have $\left.\frac{d^2y}{dx^2}\right|_{(-2,1)}=\frac{2-\frac{2(-2)^2}{1^2}}{2\cdot 1}=\frac{2-8}{2}=-3$

12. a. $x(2x-1)(3x+4) \le 0 \Rightarrow$ special points are $x=0$ or $\frac{1}{2}$ or $-\frac{4}{3}$, Then.

	$(-\infty, -\frac{4}{3})$	$(-\frac{4}{3}, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \infty)$
product	$-$	$+$	$-$	$+$
$(3x+4)$	$-$	$+$	$+$	$+$
(x)	$-$	$-$	$+$	$+$
$(2x-1)$	$-$	$-$	$-$	$+$

$\Rightarrow x \le -\frac{4}{3}$ or $0 \le x \le \frac{1}{2} \Rightarrow x \in (-\infty, -\frac{4}{3}] \cup [0, \frac{1}{2}]$.

b. $x^2 - 7x + 6 > 0 \Rightarrow (x-6)(x-1) > 0 \Rightarrow$ special points are $x = 1$ or 6

	$(-\infty, 1)$	$(1, 6)$	$(6, \infty)$
	$+$	$-$	$+$
$(x-1)$	$-$	$+$	$+$
$(x-6)$	$-$	$-$	$+$

$\Rightarrow x < 1$ or $x > 6$

$\Rightarrow x \in (-\infty, 1) \cup (6, \infty)$

13. $\frac{d}{dx}\left[(2x-5)\left[\frac{d}{dx}(2x^2+x)\right]\right] = \frac{d}{dx}\left[(2x-5)(4x+1)\right]$

chain rule

$= 2(4x+1) + (2x-5) \cdot 4 = 8x + 2 + 8x - 20 = 16x - 18$

14. Given $y^2 = 4(x+2)$ and $\left.\frac{dy}{dt}\right|_{(x,y)=(7,6)} = 3 \frac{\text{units}}{\text{second}}$.

Then do "$\frac{d}{dt}$" on both sides.

$\Rightarrow \frac{d}{dt}(y^2) = \frac{d}{dt}[4(x+2)] \Rightarrow 2y\frac{dy}{dt} = 4\frac{dx}{dt} + 0$

$\xRightarrow{\text{at } (7,6)} 2 \cdot 6 \left.\frac{dy}{dt}\right|_{(7,6)} = 4 \cdot \left.\frac{dx}{dt}\right|_{(7,6)} \Rightarrow 12 \cdot 3 = 4 \cdot \left.\frac{dx}{dt}\right|_{(7,6)} \Rightarrow \left.\frac{dx}{dt}\right|_{(7,6)} = 9 \frac{\text{units}}{\text{second}}$

15.

Let x be the distance from wall to the bottom of the ladder.
So $\frac{dx}{dt} = 6 \frac{ft}{min}$, and the height from the top of the ladder to the ground will be $y = \sqrt{15^2 - x^2}$ (by Pythagorean's thm)

Finding the rate of the descending of the man at $x=9$ means finding $\left.\frac{dy}{dt}\right|_{x=9}$.

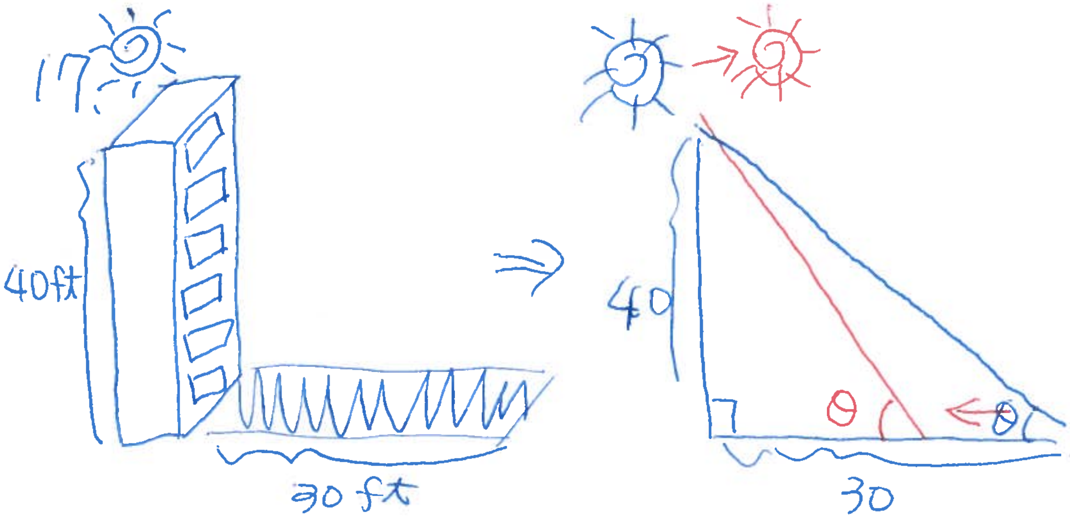
since $y = (15^2 - x^2)^{\frac{1}{2}}$, then $\frac{dy}{dt} = \frac{1}{2}(15^2 - x^2)^{-\frac{1}{2}} \cdot \left(-2x\frac{dx}{dt}\right)$

and $\left.\frac{dy}{dt}\right|_{x=9} = \frac{1}{2} \frac{1}{\sqrt{15^2 - 9^2}} \cdot (-2 \cdot 9) \cdot \underbrace{6}_{\left(\left.\frac{dx}{dt}\right|_{at\ x=9}\right)} = \frac{-54}{12} = -\frac{9}{2}$ ("$-$" means down the wall)

16. Given $y = 2x^2 + 1$ and $\frac{dy}{dt} = -2 \frac{units}{sec}$ ("$-$" means decreasing). Then finding $\left.\frac{dx}{dt}\right|_{at\ x=\frac{3}{2}}$.

do "$\frac{d}{dt}$" on both sides, $\frac{dy}{dt} = \frac{d}{dt}(2x^2+1) = 4x\frac{dx}{dt} + 0$

$\Rightarrow -2 = 4 \cdot \frac{3}{2} \cdot \left.\frac{dx}{dt}\right|_{x=\frac{3}{2}} \Rightarrow \left.\frac{dx}{dt}\right|_{x=\frac{3}{2}} = -\frac{2}{6} = -\frac{1}{3} \frac{units}{sec}$



angle is increasing at the rate $\frac{\pi}{1500}$

Let x be the ~~distance from~~ length of the shadow.

We have $\frac{d\theta}{dt} = \frac{\pi}{1500}$ and $\tan\theta = \frac{40}{x}$.

As this moment, $x=30$, we need to find $\left.\frac{dx}{dt}\right|_{x=30}$.

do "$\frac{d}{dt}$" on both sides, $\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{40}{x}\right)$, by quotient rule,

$\Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{0\cdot x - \frac{dx}{dt}\cdot 40}{x^2}$ and ~~as~~ $x = 30$, $\sec\theta = \frac{50}{40} = \frac{5}{\cancel{8}\,3}$

Then $\left(\frac{5}{\cancel{8}\,3}\right)^2 \left.\frac{d\theta}{dt}\right|_{x=30} = \frac{\cancel{30} - \left.\frac{dx}{dt}\right|_{x=30}\cdot 40}{(30)^2} \Rightarrow \frac{25}{\cancel{16}\,9}\cdot\frac{\pi}{1500}\cdot(30)^2 = \cancel{30} - 40\left.\frac{dx}{dt}\right|_{x=30}$

$\Rightarrow \left.\frac{dx}{dt}\right|_{x=30} = -\frac{1}{\cancel{40}\,8}\cdot\frac{\cancel{25}\,5}{\cancel{16}\,9}\cdot\frac{\pi}{\cancel{1500}\,\cancel{50}}\cdot\cancel{30}\cdot 3\cancel{0} = -\frac{\pi}{24}\ \frac{ft}{min}$.

("−" means decreasing).

18. Find tangent and normal line for Given Function

$\Leftrightarrow$ slope of tangent line $\frac{dy}{dx}$ and

the product of the slope of tangent and normal line is -1.

a. $y^2-x+6=0$ @ $(15,3)$

do "$\frac{d}{dx}$" on both sides, we have $\frac{d}{dx}(y^2-x+6)=\frac{d}{dx}(0)=0$

$\Rightarrow 2y\frac{dy}{dx}-1=0 \Rightarrow \frac{dy}{dx}=\frac{1}{2y} \Rightarrow$ slope at $(15,3)$: $\left.\frac{dy}{dx}\right|_{(15,3)}=\frac{1}{6}$

Tangent line: $(y-3)=\frac{1}{6}(x-15)$.

Normal line $(y-3)=-6(x-15)$.

b. $2x^2-6xy+y^2=9$ @ $(1,-1)$.

do "$\frac{d}{dx}$" on both sides, we have $\frac{d}{dx}(2x^2-6xy+y^2)=\frac{d}{dx}(9)=0$

$\Rightarrow 4x-6y-6x\frac{dy}{dx}+2y\frac{dy}{dx}=0$

@ $(1,-1)$

$\Rightarrow 4-6(-1)-6\cdot 1\cdot\left.\frac{dy}{dx}\right|_{(1,-1)}+2(-1)\left.\frac{dy}{dx}\right|_{(1,-1)}=0 \Rightarrow \left.\frac{dy}{dx}\right|_{(1,-1)}=+\frac{5}{4}$.

slope at $(1,-1)$

Tangent line: $(y+1)=\frac{5}{4}(x-1)$.

Normal line: $(y+1)=-\frac{4}{5}(x-1)$.

19. Review the intermediate value theorem (I.V.T).

If f is continuous on $[a,b]$, there is a number N such that

$$f(a) \le N \le f(b) \quad \text{or} \quad f(b) \le N \le f(a),$$

then there exists $c \in (a,b)$ such that $f(c) = N$.

Given $f(x) = 2x^5 + 3x + 1$. and interval $[-1, 2]$

To check the root of $f(x)$ means $N = 0$

Now $a = -1$, $b = 2$ and $N = 0$.

Then $f(a) = f(-1) = -2 - 3 + 1 = -4 < 0$ and

$f(b) = f(2) = 2 \cdot 2^5 + 3 \cdot 2 + 1 > 0$

$\Rightarrow f(a) < 0 < f(b)$. Thus, by I.V.T.

there is a $c \in (a,b)$ such that $f(c) = 0$.

20.

(a) $h(4)$ if $h(x) = f(g(x))$, then $h(4) = f(g(4)) \underset{g(4)=3}{=} f(3) = 2$.

(b) $h'(4)$ if $h(x) = f(g(x)) \xrightarrow[\text{rule}]{\text{chain}} h'(x) = f'(g(x)) \cdot g'(x)$.

$$h'(4) = f'(g(4)) \cdot g'(4) \underset{g(4)=3,\, g'(4)=1}{=} f'(3) \cdot 1 = 2 \cdot 1 = 2$$

(c) $h(4)$ if $g(f(x))$ $\cancel{\xrightarrow[\text{rule}]{\text{chain}} h'(x) = g'(f(x)) \cdot f'(x)}$

$$h(4) = g(f(4)) \underset{f(4)=4}{=} g(4) = 3.$$

20

(d) $h'(4)$ if $h(x) = g(f(x)) \xrightarrow[\text{rule}]{\text{chain}} h'(x) = g'(f(x)) \cdot f'(x)$, then

$$h'(4) = g'(f(4)) \cdot f'(4) = g'(4) \cdot 3 = 1 \cdot 3 = 3$$

(e) $h'(4)$ if $h(x) = \frac{g(x)}{f(x)}$, $h'(x) = \frac{g'(x)f(x) - f'(x)g(x)}{[f(x)]^2}$, then

$$h'(4) = \frac{g'(4)f(4) - f'(4)g(4)}{[f(4)]^2} = \frac{1 \cdot 4 - 3 \cdot 3}{16} = -\frac{5}{16}.$$

(f) $h'(4)$ if $h(x) = f(x)g(x)$. $h'(x) = f'(x)g(x) + f(x)g'(x)$.

$$h'(4) = f'(4)g(4) + f(4)g'(4) = 3 \cdot 3 + 4 \cdot 1 = 13.$$

21. See the graph.

22.

8.