Math 1450, Honor Calculus Practice 3, Fall 2015.

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1. Find the limit $\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)$.

$$= \lim_{x\to\infty} \frac{(\sqrt{9x^2+x}-3x)(\sqrt{9x^2+x}+3x)}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{x\cdot\frac{1}{x}}{\frac{1}{x}\cdot(\sqrt{9x^2+x}+3x)} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{9x^2+x}{x^2}}+3} = \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{6}$$

2. Find $\lim_{x\to\infty} f(x)$ if , for all $x > 1$, $\frac{10e^x-21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x-1}}$.

By Squeeze theorem, we have

$$\lim_{x\to\infty} \frac{10e^x-21}{2e^x} = \lim_{x\to\infty} \frac{(10e^x-21)\frac{1}{e^x}}{2e^x\frac{1}{e^x}} = \lim_{x\to\infty} \frac{10-\frac{21}{e^x}}{2} = \frac{10}{2} = 5 \text{ and}$$

$$\lim_{x\to\infty} \frac{5\sqrt{x}}{\sqrt{x-1}} = \lim_{x\to\infty} \frac{5\sqrt{x}\cdot\frac{1}{\sqrt{x}}}{\sqrt{x-1}\cdot\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{5}{\sqrt{\frac{x-1}{x}}} = \lim_{x\to\infty} \frac{5}{\sqrt{1-\frac{1}{x}}} = 5.$$

Then $\lim_{x\to\infty} f(x) = 5$.

3. Determine whether $f'(0)$ exists if $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$

By def. of derivative, we have

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0^+} h \sin\left(\frac{1}{h}\right) \overset{\text{by Squeeze}}{=} 0$$

$0+h>0$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0^-} h \sin\left(\frac{1}{h}\right) \overset{\text{Squeeze}}{=} 0$$

$0+h<0$

4. (a) Find the limit $\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}$. (b) Using (a), find the limit $\lim_{x\to 0}(1+x)^{\frac{3}{x}}$.

(a) Let $f(x) = \ln x$. $f'(x) = \frac{1}{x}$. Then

$$f'(1) = \lim_{x\to 0} \frac{f(1+x)-f(1)}{x} = \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x\to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x\to 0} \frac{1}{x}\ln(1+x) = \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}.$$

$$\Rightarrow \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = f'(1) = \frac{1}{1} = 1$$

(b)

$$\lim_{x\to 0} (1+x)^{\frac{3}{x}} = \lim_{x\to 0} e^{\ln (1+x)^{\frac{3}{x}}} = \lim_{x\to 0} e^{3\ln(1+x)^{\frac{1}{x}}}$$

e^x is continuous and $\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}$ exists by (a)

$$\overset{\downarrow}{=} e^{\lim_{x\to 0} 3\ln(1+x)^{\frac{1}{x}}} = e^{3\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}} = e^{3\cdot 1} = e^3.$$