Honor Calculus, Math 1450 – Assignment 8 – Solutions

(1) <u>§11.4</u>

20. $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$.

let $a_n = \frac{n+4^n}{n+6^n}$. since $n < 4^n$ for $n \in \mathbb{N}$ and

$\frac{1}{n+6^n} < \frac{1}{6^n}$, we have $a_n < \frac{4^n+4^n}{6^n} = 2\left(\frac{4}{6}\right)^n$

By the Comparison Test, since $\frac{4}{6} < 1$, $\sum 2 \cdot \left(\frac{4}{6}\right)^n$ converges

and $\sum \frac{n+4^n}{n+6^n} < \sum 2 \cdot \left(\frac{4}{6}\right)^n$, then $\sum \frac{n+4^n}{n+6^n}$ converges.

24. $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$

let $a_n = \frac{n^2-5n}{n^3+n+1}$ and $b_n = \frac{1}{n}$. We have

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and $\sum b_n$ diverges. Then

by The Limit Comparison Test, we have $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ diverges.

39. Assume $a_n \geq 0$ and $\sum a_n$ converges. It implies

$a_n \to 0$ as $n \to \infty$, there exists an integer $N > 0$ such that

$a_n < 1$ for $n \geq N$. Thus, $a_n^2 \leq a_n < 1$ for $n \geq N$.

Then, by the comparison Test, since $\sum a_n$ converges,

$\sum a_n^2$ also converges.

§11.4

40. (a) Suppose $\sum a_n$, $\sum b_n$ are series with $a_n > 0$, $b_n > 0$ $\forall n \in \mathbb{N}$ and $\sum b_n$ is convergent. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then there is an integer $N > 0$ such that

$$\frac{a_n}{b_n} < 1 \text{ as } n > N. \text{ Thus } a_n < b_n \text{ as } n > N.$$

Since $\sum b_n$ converges, by the comparison Test, $\sum a_n$ also converges.

(b) (i) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

let $a_n = \frac{\ln(n)}{n^3}$ and $b_n = \frac{1}{n^2}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n)}{n} \underset{(\frac{\infty}{\infty})}{\overset{(L')}{=}} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0.$$ By part (a)

Since $\sum b_n = \sum \frac{1}{n^2}$ converges (by p-series), we have $\sum a_n = \sum \frac{\ln(n)}{n^3}$ converges.

(ii) $\sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}e^n}$

let $a_n = \frac{\ln(n)}{\sqrt{n}e^n}$, $b_n = \frac{\ln(n)}{n^3}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3} \to n^{\frac{5}{2}}}{\sqrt{n}e^n} \underset{(\frac{\infty}{\infty})}{\overset{(L')}{=}} \lim_{n\to\infty} \frac{\frac{5}{2}n^{\frac{3}{2}}}{e^n} \underset{(\frac{\infty}{\infty})}{\overset{(L')}{=}} \lim_{n\to\infty} \frac{\frac{15}{4}n^{\frac{1}{2}}}{e^n}$$

$$\underset{(\frac{\infty}{\infty})}{\overset{(L')}{=}} \lim_{n\to\infty} \frac{\frac{15}{8}}{n^{\frac{1}{2}}e^n} = 0.$$

Since, by (i), $\sum \frac{\ln(n)}{n^3}$ converges, then, by (a), $\sum \frac{\ln(n)}{\sqrt{n}e^n}$ converges.

§11.4

42. Let $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$. We have $\Sigma b_n = \Sigma \frac{1}{n}$ diverges

and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$ but

$\Sigma a_n = \Sigma \frac{1}{n^2}$ converges.

(So comparing with 40, if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. we only have Σb_n converges $\Rightarrow \Sigma a_n$ converges, BUT "$\Leftarrow$" is WRONG!)

§11.5

4. $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

Let $b_n = \frac{1}{\sqrt{n}}$, we have ① $b_n \to 0$ as $n \to \infty$ ② $b_n > b_{n+1} \geq 0$ for all n.

Then by The Alternating Series Test, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$

Let $b_n = \frac{1}{\ln(n+4)}$, we have ① $b_n > b_{n+1} \geq 0$ for all $n \in \mathbb{N}$ and ② $\lim_{n\to\infty} b_n = 0$. Then

By Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$ converges.

§11.5

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n!} = \frac{1}{1!} + \frac{0}{2!} + \frac{-1}{3!} + \frac{0}{4!} + \frac{1}{5!} + \frac{0}{6!} - \frac{1}{7!} + \frac{0}{8!} + \cdots$

$$\sin(\tfrac{n\pi}{2}) = \begin{cases} 1, & n = 4k+1, \ k \in \mathbb{N}; \\ -1, & n = 4k+3, \ k \in \mathbb{N}; \\ 0, & n = 4k+2, 4k, \ k \in \mathbb{N} \end{cases}$$

$$= \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{(2l-1)!} \quad \text{or} \quad \sum_{l=0}^{\infty} \frac{(-1)^{l}}{(2l+1)!}$$

Let $b_l = \frac{1}{(2l-1)!}$. We have ① $\lim_{l\to\infty} b_l = 0$ and

② $b_l > b_{l+1} > 0 \ \forall l \in \mathbb{N}$. Then, by The Alternating Series Test,

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n!} = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{(2l-1)!} \text{ converges.}$$

34. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$, By The Alternating Series Test,

We want to find p such that ① $\frac{(\ln(n))^p}{n} \to 0$ as $n \to \infty$

and ② $\frac{(\ln(n))^p}{n} \geq \frac{(\ln(n+1))^p}{n+1} > 0, \ \forall n \in \mathbb{N}$.

For ①, $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^p}{n} = \lim_{n\to\infty} \frac{p\cdot\frac{1}{n}(\ln n)^{p-1}}{1} \underset{(\frac{\infty}{\infty})}{\overset{(L')}{=}} \lim_{n\to\infty} p(p-1)\frac{(\ln n)^{p-2}}{n} = \cdots$

$\Rightarrow p$ can be any number.

For ②, consider $\displaystyle\left(\frac{(\ln x)^p}{x}\right)' = \frac{p\frac{(\ln x)^{p-1}}{x}\cdot x - (\ln x)^p}{x^2} = \frac{(\ln x)^{p-1}[p - \ln x]}{x^2} < 0$
$\forall x \geq 2$

$\Rightarrow p - \ln x < 0 \Rightarrow p < \ln x. \ \forall x \geq 2 \Rightarrow p \leq \ln 2$,

§11.6

10. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

I. For absolutely convergent, conside $\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{\sqrt{n^3+2}} \right| = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$,

let $a_n = \frac{n}{\sqrt{n^3+2}}$, $b_n = \frac{1}{n^{\frac{1}{2}}}$, we have $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and $\sum b_n$ diverges, (by p-series Test)

So $\sum \frac{n}{\sqrt{n^3+2}}$ Diverges $\Rightarrow$ NOT absolutely convergent.

II. For conditionally convergent, let $b_n = \frac{n}{\sqrt{n^3+2}}$, we have

① $\lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = 0$ and ② consider $\left(\frac{x}{\sqrt{x^3+2}}\right)' = \frac{\sqrt{x^3+2} - x \cdot \frac{3x^2}{2} \cdot \frac{1}{\sqrt{x^3+2}}}{(x^3+2)}$ < 0 $\cancel{>}0$

$\left(= \frac{x^3+2-\frac{3}{2}x^3}{(x^3+2)\sqrt{x^3+2}} \right) \forall x > 0$.

So $\frac{x}{\sqrt{x^3+2}}$ is ~~an increasing~~ a decreasing function which implies ~~$b_{n+1} > b_n$~~ $b_n > b_{n+1}$

$\Rightarrow \sum (-1)^n \frac{n}{\sqrt{n^3+2}}$ is ~~NOT~~ conditionally convergent ~~and it is divergent.~~

14. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$

For absolutely convergent, consider $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$.

let $a_n = \frac{n^2 2^n}{n!}$, we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 2^{n+1}}{(n+1)!} \frac{n!}{n^2 2^n}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{1}{n+1} \cdot 2 = 0 < 1$.

So, by the Ratio test, $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ converges

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ is absolutely convergent.

§ 11.6

16. $\sum_{n=1}^{\infty} \frac{3-\cos(n)}{n^{\frac{2}{3}}-2}$

Let $a_n = \frac{3-\cos(n)}{n^{\frac{2}{3}}-2}$, $b_n = \frac{2}{n^{\frac{2}{3}}}$ and $b_n < a_n$, since $\sum b_n$ diverges,

Then $\sum a_n = \sum \frac{3-\cos(n)}{n^{\frac{2}{3}}-2}$ diverges.

20. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^n}$

For absolutely convergent, consider $\sum_{n=1}^{\infty} \left|(-1)^n \cdot \left(\frac{2}{n}\right)^n\right| = \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$.

By root test, let $a_n = \left(\frac{2}{n}\right)^n$, we have.

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{(-2)^n}{n}$ is abs. convergent.

22. $\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n} = \sum_{n=2}^{\infty} \frac{(-1)^{5n}(2n)^{5n}}{(n+1)^{5n}}$

① For abs. convergent, consider $\sum_{n=2}^{\infty} \left|\frac{(-1)^{5n}(2n)^{5n}}{(n+1)^{5n}}\right| = \sum_{n=2}^{\infty} \frac{(2n)^{5n}}{(n+1)^{5n}}$

By Root test, let $a_n = \left(\frac{2n}{n+1}\right)^{5n}$, we have.

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^5 = 2^5 > 1 \Rightarrow \sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ is NOT abs. convergent.

② For cond. convergent, let $b_n = \left(\frac{2n}{n+1}\right)^{5n}$

since $b_n \not\to 0$ as $n \to \infty$, so $\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ diverges.

Or, since $\left(\frac{-2n}{n+1}\right)^{5n} \not\to 0$ as $n\to\infty$. By Basic Divergent Test,

$\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ diverges.

30\. Assume $a_1 = 1$. $a_{n+1} = \frac{2+\cos(n)}{\sqrt{n}} a_n$, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2+\cos(n)}{\sqrt{n}} = 0 < 1. \text{ So,}$$

by ~~Root~~ Ratio Test, we obtain $\sum a_n$ converges.

31\.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$, Let $a_n = \frac{1}{n^3}$, Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} = 1 \Rightarrow$ Fail!

(b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Let $a_n = \frac{n}{2^n}$, Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$

$\Rightarrow$ convergent!

(c) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$, Let $a_n = \frac{(-3)^{n-1}}{\sqrt{n}}$, Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{3^{n-1}} = 3 > 1$

$\Rightarrow$ Divergent!

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$. Let $a_n = \frac{\sqrt{n}}{1+n^2}$, Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} = 1$

$\Rightarrow$ Fail!

34\. Let $\sum a_n$ be a series with $a_n > 0$, and let $r_n = \frac{a_{n+1}}{a_n}$

Suppose $\lim_{n\to\infty} r_n = L < 1$, Let $R_n = a_{n+1} + a_{n+2} + \cdots$.

(a) Since $r_n = \frac{a_{n+1}}{a_n}$, we have $a_{n+1} = r_n a_n = r_n \cdot r_{n-1} a_{n-1} = r_n r_{n-1} r_{n-2} a_{n-2}$
$= \cdots$ and

$$R_n = a_{n+1} + a_{n+1} r_{n+1} + a_{n+1} r_{n+1} r_{n+2} + a_{n+1} r_{n+1} r_{n+2} r_{n+3} + \cdots$$

$$\le a_{n+1} + a_{n+1} r_{n+1} + a_{n+1} r_{n+1}^2 + a_{n+1} r_{n+1}^3 + \cdots$$

($\{r_n\}$ is decreasing means $r_n > r_N$ for $N > n$)

$$= a_{n+1}(1 + r_{n+1} + r_{n+1}^2 + \cdots)$$

$$\underset{r_{n+1}<1}{=} a_{n+1} \cdot \frac{1}{1 - r_{n+1}}$$

34.

(b). Similarly, we have

$$R_n = a_{n+1} + a_{n+1} r_{n+1} + a_{n+1} r_{n+1} r_{n+2} + a_{n+1} r_{n+1} r_{n+2} r_{n+3} + \cdots$$

$$\leq a_{n+1} + a_{n+1} L_1 + a_{n+1} L_1^2 + \cdots = a_{n+1} \frac{1}{1-L_1}.$$

(since $r_n \to L_1$ as $n \to \infty$, and $\{r_n\}$ is increasing $\Rightarrow r_n < L_1 < 1$ $\forall n \in \mathbb{N}$)

(4)

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\log(n+2)}$.

Let $b_n = \dfrac{1}{\log(n+2)}$. ① since $b_n \to 0$ as $n \to \infty$, and

② $b_n > b_{n+1} > 0$ $\forall n \in \mathbb{N}$, By The Alternating Series Test,

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\log(n+2)}$ converges.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{n^{\frac{1}{2}}+7}{n^2-2n}$

Let $a_n = \dfrac{n^{\frac{1}{2}}+7}{n^2-2n}$, $b_n = \dfrac{1}{n^{\frac{3}{2}}}$, we have $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$.

By The Limit Comparison Test,

Since $\sum b_n = \sum \dfrac{1}{n^{\frac{3}{2}}}$ converges, $\sum a_n = \sum \dfrac{n^{\frac{1}{2}}+7}{n^2-2n}$ converges.

(4)

(c) $\sum_{n=1}^{\infty} \frac{3n^2+n+1}{(\sqrt{2})^n}$. let $a_n = \frac{3n^2+n+1}{(\sqrt{2})^n}$,

By Root Test, we have $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{3n^2+n+1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} < 1$.

$$\left(\lim_{n\to\infty} \sqrt[n]{3n^2+n+1} = e^{\lim_{n\to\infty} \ln \sqrt[n]{3n^2+n+1}} = e^{\lim_{n\to\infty} \frac{\ln(3n^2+n+1)}{n}} \overset{(L')}{=} e^{\lim_{n\to\infty} \frac{6n+1}{3n^2+n+1}} = e^0 = 1 \right)$$

Thus. $\sum_{n=1}^{\infty} \frac{3n^2+n+1}{(\sqrt{2})^n}$ is Convergent.

(d) $\sum_{n=1}^{\infty} \frac{n!}{2^{2^n}}$. let $a_n = \frac{n!}{2^{2^n}}$, we have..

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{2^{2^{n+1}}} \cdot \frac{2^{2^n}}{n!} = \lim_{n\to\infty} n+1 \cdot \frac{1}{2^{2^n}} = 0 < 1.$$

So, by Ratio test, $\sum_{n=1}^{\infty} \frac{n!}{2^{2^n}}$ converges.

(5) let $a_n = \frac{(x-2)^n}{n3^n}$, By Root test, $\sum a_n$ converges $\Leftrightarrow$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-2|}{\sqrt[n]{n} \cdot 3} = \frac{|x-2|}{3} < 1.$$

$$\Rightarrow 3 < x-2 < 3 \Rightarrow -1 < x < 5.$$

Checking Two end points.

As $x=-1$, We have $a_n = \frac{(-3)^n}{n(3^n)} = \frac{(-1)^n}{n}$ and $\sum a_n$ converges by Alternating Test.

As $x=5$. We have $a_n = \frac{(3)^n}{n(3^n)} = \frac{1}{n}$, and $\sum a_n$ Diverges by P-series Test

$\Rightarrow \sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}$ Converges if $-1 \le x < 5$.

(6) let $a_n = \frac{x^n}{n!}$. By Ratio Test,

$\sum a_n$ converges $\Leftrightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = 0 < 1$

$\Rightarrow$ x can be any number.