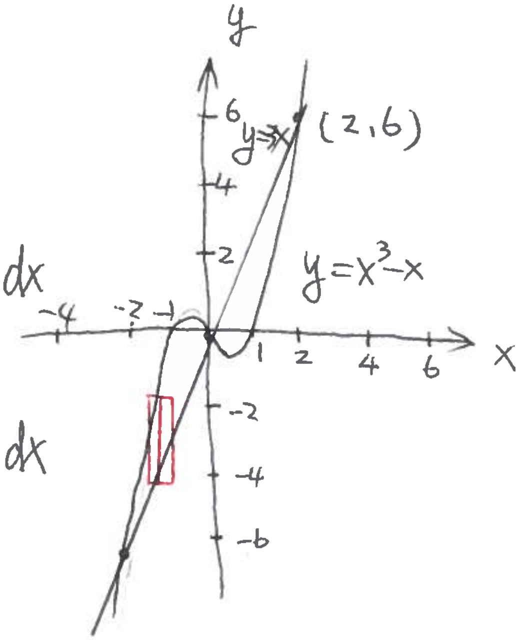
Honor Calculus, Math 1450 - Assignment 6 - Solution

§6.1

(16) Given $y=x^3-x$ and $y=3x$.



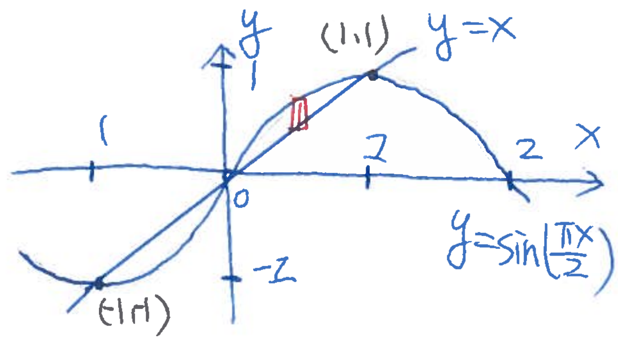
$$\text{Area}=\int_0^2\left[3x-(x^3-x)\right]dx+\int_{-2}^{0}\left[(x^3-x)-3x\right]dx$$

$$=\int_0^2(3x-x^3+x)\,dx+\int_{-2}^{0}(x^3-x-3x)\,dx$$

$$=\left[\frac{3}{2}x^2-\frac{x^4}{4}+\frac{x^2}{2}\Big|_0^2\right]\times 2$$

$$=\left(\frac{12}{2}-4+2\right)\times 2=8.$$

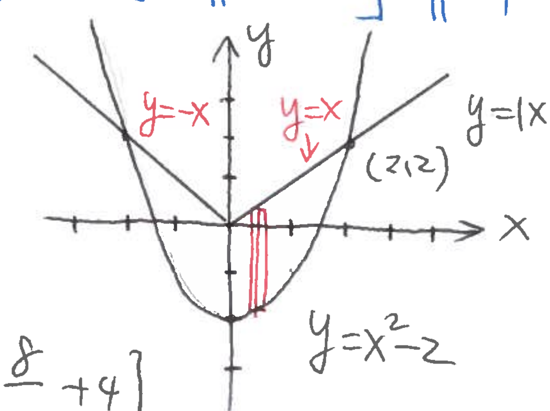
(22) Given $y=\sin\left(\frac{\pi x}{2}\right)$ and $y=x$



$$\text{Area}=2\cdot\int_0^1\left(\sin\left(\frac{\pi x}{2}\right)-x\right)dx$$

$$=2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)-\frac{x^2}{2}\right]_0^1=-2\left[\frac{2}{\pi}\cdot 0+\frac{1}{2}-\frac{2}{\pi}\cdot 1-0\right]=\frac{4}{\pi}-1$$

(26) Given $y=|x|$. and $y=x^2-2$



$$\text{Area}=2\cdot\int_0^2\left[x-(x^2-2)\right]dx$$

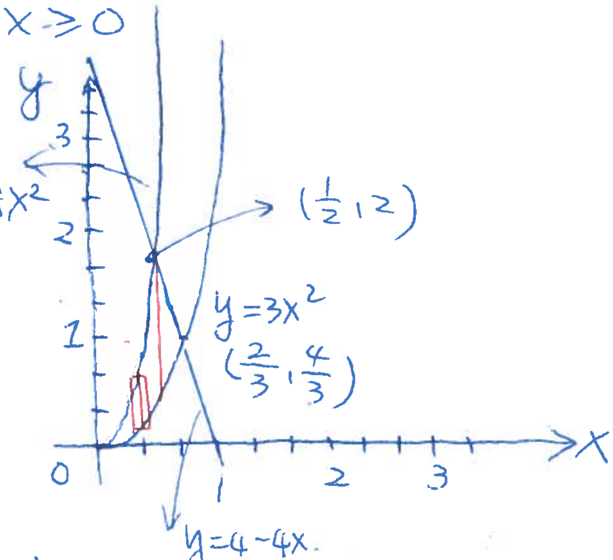
$$=2\cdot\left[\frac{x^2}{2}-\frac{x^3}{3}+2x\right]_0^2=2\cdot\left[\frac{4}{2}-\frac{8}{3}+4\right]$$

$$=\frac{20}{3}$$

§6.1

(28) Given $y=3x^2$, $y=8x^2$, $4x+y=4$, $x\geq 0$

$\Downarrow$ $y=4-4x$



(intersection points:

$4-4x=3x^2 \Rightarrow 3x^2+4x-4=0$

$\Rightarrow (x+2)(3x-2)=0 \Rightarrow x=\frac{2}{3}$, $y=\frac{4}{3}$

$4-4x=8x^2 \Rightarrow 8x^2+4x-4=0 \Rightarrow 2x^2+x-1=0$

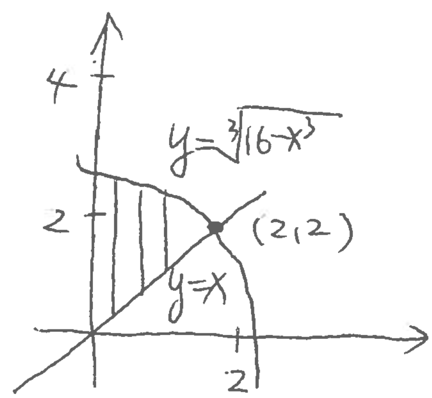
$\Rightarrow (x+1)(2x-1)=0 \Rightarrow x=\frac{1}{2}$, $y=2$)

$$\text{Area}=\int_0^{\frac{1}{2}}(8x^2-3x^2)\,dx+\int_{\frac{1}{2}}^{\frac{2}{3}}(4-4x-3x^2)\,dx$$

$$=\frac{5}{3}x^3\Big|_0^{\frac{1}{2}}+4x-2x^2-x^3\Big|_{\frac{1}{2}}^{\frac{2}{3}}=\frac{5}{24}+4\left(\frac{2}{3}-\frac{1}{2}\right)-2\left(\frac{4}{9}-\frac{1}{4}\right)-\left(\frac{8}{27}-\frac{1}{8}\right)$$

$$=\frac{5}{24}+\frac{4}{6}-2\cdot\frac{7}{36}-\frac{37}{216}=\frac{68}{216}=\frac{17}{54}$$

(34) Given $y=\sqrt[3]{16-x^3}$ and $y=x$, $x=0$



$n=4$, partition $P=\{\overset{x_0}{0},\overset{x_1}{\frac{1}{2}},\overset{x_2}{1},\overset{x_3}{\frac{3}{2}},\overset{x_4}{2}\}$ of $[0,2]$

mid-point: $\frac{1}{4}$ $\frac{3}{4}$ $\frac{5}{4}$ $\frac{7}{4}$

$$\text{Area}=\int_0^2 \sqrt[3]{16-x^3}-x\,dx$$, let $f(x)=\sqrt[3]{16-x^3}-x$.

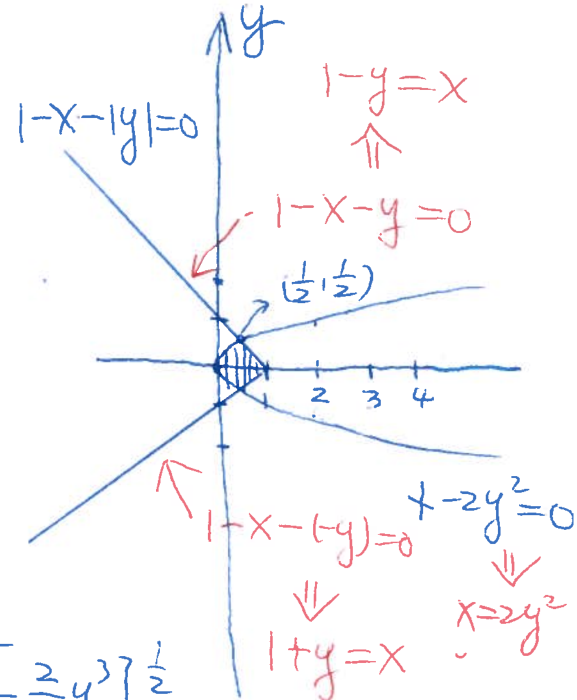
$$\text{Riemann Sum of Area}=\sum_{i=0}^{3} f\left(\frac{x_{i+1}-x_i}{2}\right)\cdot[x_{i+1}-x_i]$$

$$=f\left(\tfrac{1}{4}\right)\cdot\tfrac{1}{2}+f\left(\tfrac{3}{4}\right)\cdot\tfrac{1}{2}+f\left(\tfrac{5}{4}\right)\tfrac{1}{2}+f\left(\tfrac{7}{4}\right)\cdot\tfrac{1}{2}$$

36.1

40. Given $x-2y^2 \geq 0$, $1-x-|y| \geq 0$.

(consider $x=2y^2$ and $|y| = 1-x$.)



(intersection: $1-y=2y^2$

$\Rightarrow 2y^2+y-1=0 \Rightarrow (y+1)(2y-1)=0$

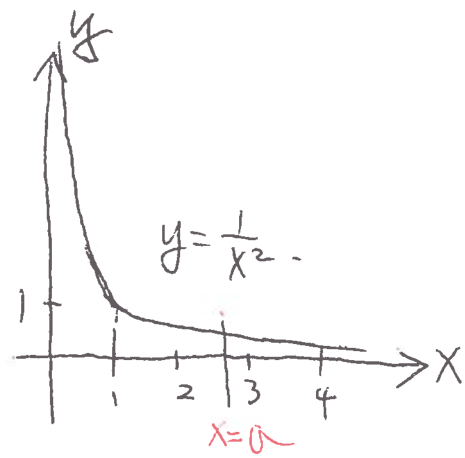
$\Rightarrow y=\frac{1}{2},\ x=\frac{1}{2}$)

$$\text{Area} = 2\cdot\int_0^{\frac{1}{2}} 1-y-2y^2\,dy = 2\left[y-\frac{y^2}{2}-\frac{2}{3}y^3\right]_0^{\frac{1}{2}}$$

$$= 2\left[\frac{1}{2}-\frac{1}{8}-\frac{2}{3}\cdot\frac{1}{8}\right] = 2\cdot\frac{7}{24} = \underline{\frac{7}{12}}$$

50. (a) Find a such that

$$\int_1^a \frac{dx}{x^2} = \int_a^4 \frac{dx}{x^2}$$



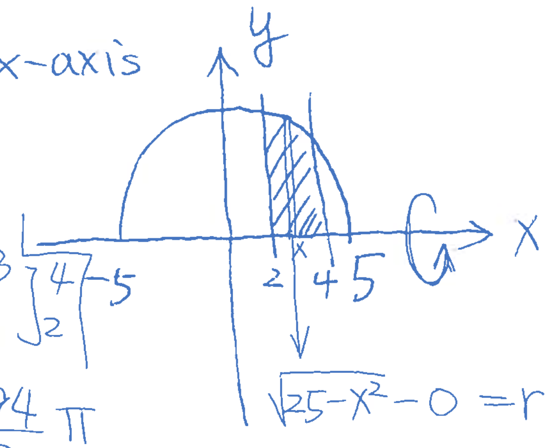
$$\Rightarrow +\frac{1}{x}\Big|_1^a = +\frac{1}{x}\Big|_a^4$$

$$\frac{1}{a}-1 = \frac{1}{4}-\frac{1}{a}$$

$$\Rightarrow \frac{2}{a} = \frac{5}{4} \Rightarrow \underline{a=\frac{8}{5}}$$

§6.2

(4) Given $y=\sqrt{25-x^2}$, $y=0$, $x=2$, $x=4$, and find the volume by rotating about x-axis

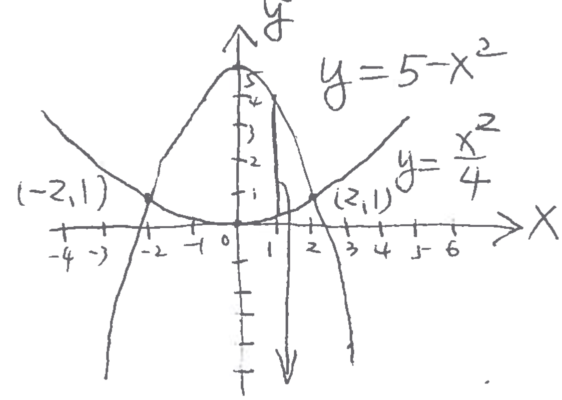


$$\text{Volume}=\int_2^4 \pi r(x)^2\,dx$$

$$=\pi\int_2^4 \left(\sqrt{25-x^2}\right)^2 dx=\pi\left[25x-\frac{x^3}{3}\right]_2^4$$

$$=\pi\left[25(4-2)-\frac{1}{3}(4^3-2^3)\right]=\underline{\frac{94}{3}\pi}$$

(8) Given $y=\frac{x^2}{4}$, $y=5-x^2$ and find the volume by rotating about x-axis.



$$V=\pi\int_{-2}^{2} R(x)^2-r(x)^2\,dx$$

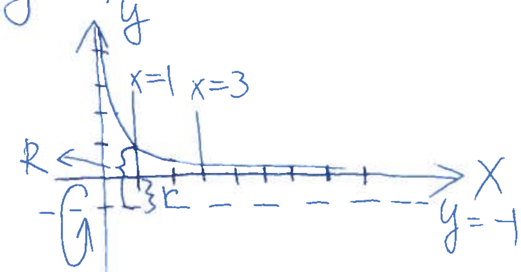
$$=2\pi\int_0^2 (5-x^2)^2-\left(\frac{x^2}{4}\right)^2 dx$$

$$=2\pi\left[25x-\frac{10}{3}x^3+\frac{x^5}{5}-\frac{x^5}{80}\right]_0^2$$

$R=5-x^2$, $r=\frac{x^2}{4}$

$$=2\pi\left[25(2-0)-\frac{10}{3}(8)+\frac{\cancel{15}^{3}}{\cancel{80}\,\cancel{16}}\cdot\cancel{32}^{2}\right]=2\pi\,\frac{76}{3}=\underline{\frac{152}{3}\pi}$$

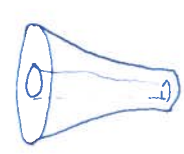
(14) Given $y=\frac{1}{x}$, $y=0$, $x=1$, $x=3$, about $y=-1$



$$V=\pi\int_1^3 R(x)^2-r(x)^2\,dx=\pi\int_1^3 \left(\frac{1}{x}+1\right)^2-1\,dx$$

$$=\pi\left[-\frac{1}{x}+2\ln x+x-x\right]_1^3$$

$$=\pi\left[-\left(\frac{1}{3}-1\right)+2\ln 3-2\ln 1\right]$$

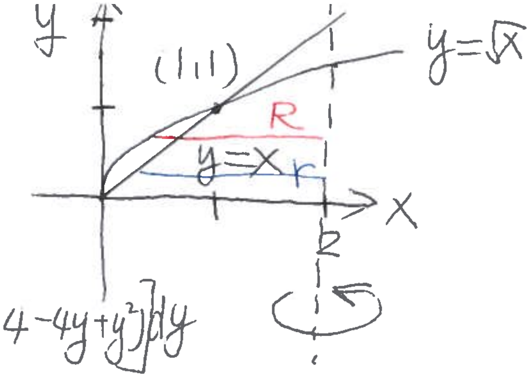


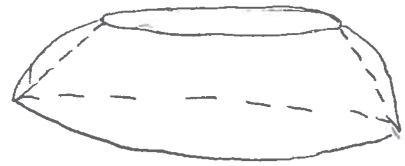
$R=\frac{1}{x}+1$, $r=1$

$$=\underline{\left(\frac{2}{3}+2\ln 3\right)\pi}$$

§6.2

(16) Given $y=x$, $y=\sqrt{x}$, about $x=2$.





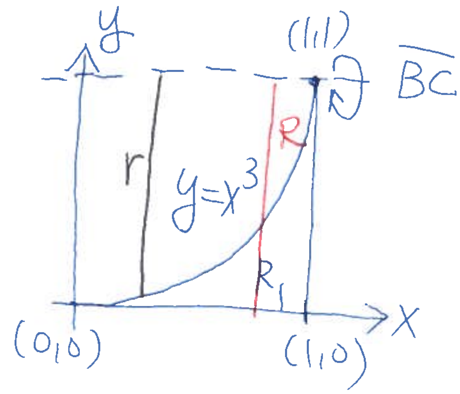
$R = 2-y^2$, $r = 2-y$

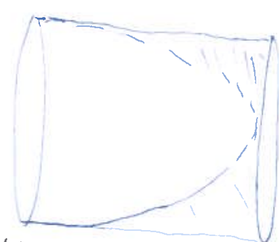
$V = \pi \int_0^1 R^2(y) - r^2(y)\, dy = \pi \int_0^1 [4-4y^2+y^4-(4-4y+y^2)]\,dy$

$= \pi \int_0^1 (4y-5y^2+y^4)\,dy$

$= \pi \left[2y^2 - \frac{5}{3}y^3 + \frac{y^5}{5}\right]_0^1 = \pi\left[2-\frac{5}{3}+\frac{1}{5}\right] = \frac{8}{15}\pi$

(22) $R=1$, $r=1-x^3$



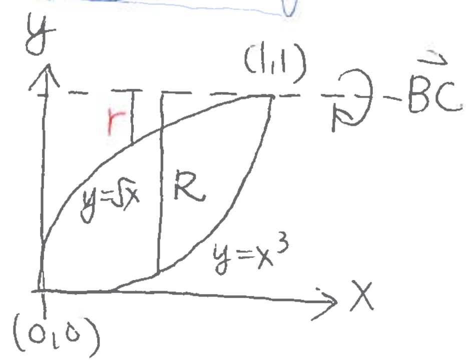


$V=\pi\int_0^1 R^2(x)-r^2(x)\,dx = \pi\int_0^1 1-(1-2x^3+x^6)\,dx$

$= \pi\left[\frac{x^4}{2}-\frac{x^7}{7}\right]_0^1 = \frac{5}{14}\pi$

(30)

$R(x) = 1-x^3$, $r(x) = 1-\sqrt{x}$



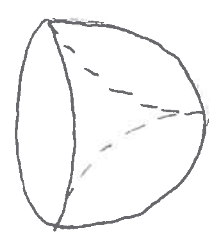
$V = \pi\int_0^1 R^2(x) - r^2(x)\,dx = \pi\int_0^1 (1-x^3)^2-(1-\sqrt{x})^2\,dx$

$= \pi\int_0^1 1-2x^3+x^6-(1-2\sqrt{x}+x)\,dx$

$= \pi\int_0^1 2\sqrt{x}-x-2x^3+x^6\,dx$

$= \pi\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{x^2}{2}-\frac{x^4}{2}+\frac{x^7}{7}\right]_0^1$

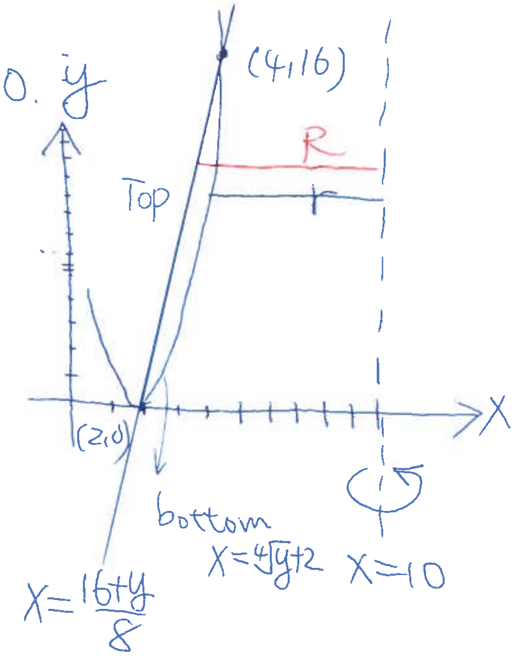
$= \pi\,\frac{10}{21} = \frac{10\pi}{21}$



§6.2

(32) Given $y=(x-2)^4$, $8x-y=16$, about $x=10$.

$\Rightarrow y=8x-16$



Intersection Points of two curves:

$$(x-2)^4 = 8x-16 \Rightarrow (x-2)\left[(x-2)^3-8\right]=0$$

$$\Rightarrow (x-2)(x-2-2)\left[(x-2)^2+2(x-2)+4\right]=0$$

$$\Rightarrow (x-2)(x-4)(x^2-2x+4)=0$$

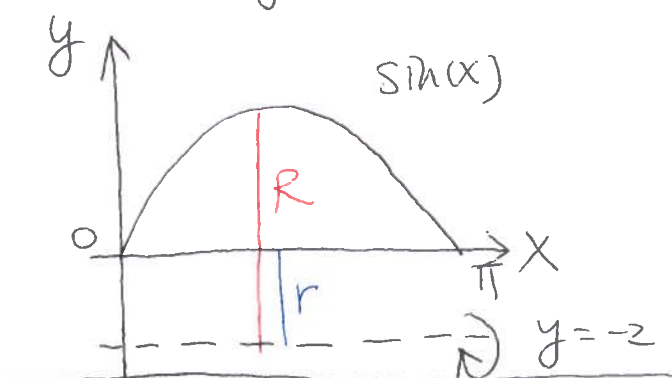
$\Rightarrow x=2$ or 4. $\Rightarrow$ points $(2,0)$, $(4,16)$

$$R=10-\frac{16+y}{8}=8-\frac{y}{8}, \quad r=10-(\sqrt[4]{y}+2)$$
$$=8-\sqrt[4]{y}$$

$$V=\pi\int_0^{16} R(y)^2-r(y)^2\,dy=\pi\int_0^{16}\left(8-\frac{y}{8}\right)^2-\left(8-\sqrt[4]{y}\right)^2 dy.$$

(34) Give $y=0$, $y=\sin(x)$, $0\le x\le\pi$, about $y=-2$

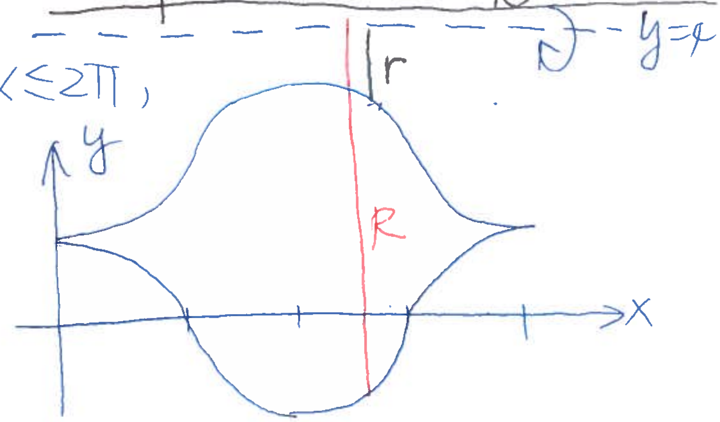
$R=\sin(x)-(-2)$, $r=2$



$$V=\pi\int_0^{\pi} R^2(x)-r^2(x)\,dx$$
$$=\pi\int_0^{\pi}\left(\sin(x)+2\right)^2-4\,dx$$

(36) Given $y=\cos(x)$, $y=2-\cos(x)$, $0\le x\le 2\pi$,

about $y=4$.

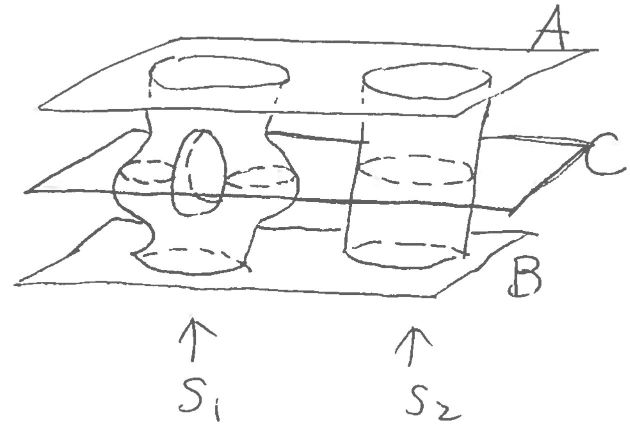


$$R(x)=4-\cos(x), \quad r(x)=4-(2-\cos(x))$$
$$=2+\cos(x)$$

$$V=\pi\int_0^{2\pi}R^2(x)-r^2(x)\,dx=\pi\int_0^{2\pi}(4-\cos(x))^2-(2+\cos(x))^2\,dx.$$

§6.2

(65) (a) See the picture.



There are two solids S_1, S_2 between two parallel planes A, B.

For an arbitrary plane C which is parallel with A & B, the intersection areas of S_1 and S_2 on C is equal with each other, namely,

$$\text{area of } S_1|_{C} = \text{area of } S_2|_{C}.$$

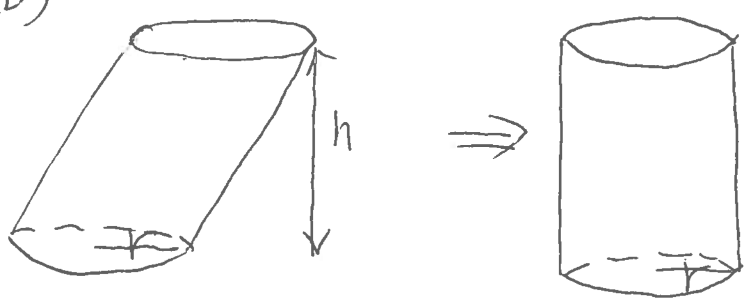
Then, since C is arbitrary, we have

$$\text{area of } S_1|_{\text{parallel plane between } A \& B} = \text{area of } S_2|_{\text{parallel plane between } A \text{ and } B}$$

Thus, let $\gamma = \{\text{the parallel planes between } A \& B\}$

$$\text{Volume of } S_1 = \sum_{C \in \gamma} \text{area of } S_1|_C = \sum_{C\in\gamma} \text{area of } S_2|_C = \text{Volume of } S_2.$$

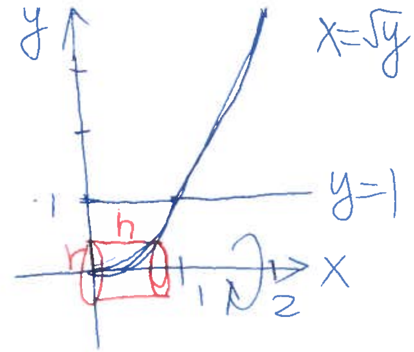
(b)



$\Rightarrow$ Volume $= \pi r^2 h$.

§6.3

(10) Given $x=\sqrt{y}$, $x=0$, $y=1$, about x-axis.



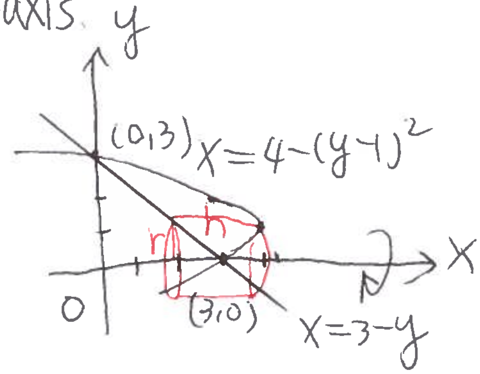
$h=\sqrt{y}-0$, $r=y$

By shell method we have

$$V=\int_0^1 2\pi r(y)h(y)\,dy = 2\pi\int_0^1 y\sqrt{y}\,dy \quad (\nearrow y^{\frac{3}{2}})$$

$$=2\pi\cdot\frac{2}{5}y^{\frac{5}{2}}\Big|_0^1=\underline{\frac{4}{5}\pi}$$

(14) Given $\underbrace{x+y=3}_{x=3-y}$, $x=4-(y-1)^2$, about x-axis.



Intersection points. $3-y=4-(y-1)^2$

$$\Rightarrow y^2-2y+1-4+3-y=0 \Rightarrow y^2-3y=0$$

$$y=0 \text{ or } 3 \Rightarrow (3,0),\ (0,3)$$

$h=4-(y-1)^2-(3-y)$

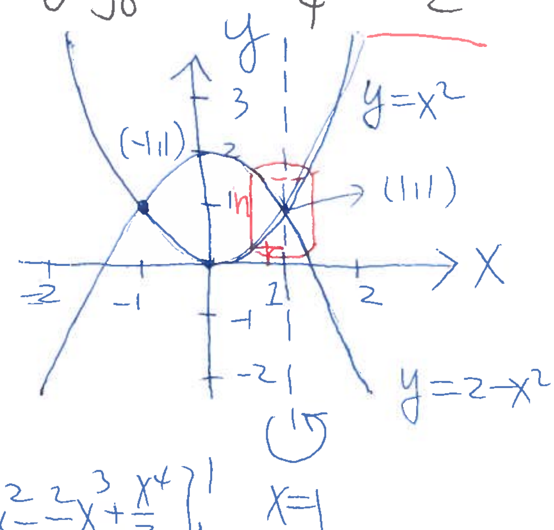
$r=y$

$$V=\int_0^3 2\pi r(y)\cdot h(y)\,dy$$

$$=2\pi\int_0^3 y\cdot\left(4-(y^2-2y+1)-3+y\right)dy$$

$$=2\pi\int_0^3 y\cdot(-y^2+3y)\,dy=2\pi\left[-\frac{y^4}{4}+y^3\right]_0^3=2\pi\frac{27}{4}=\underline{\frac{27\pi}{2}}$$

(18) Given $y=x^2$, $y=2-x^2$, about $x=1$



$h=2-x^2-x^2=2-2x^2$, $r=1-x$

$-1\le x\le 1$

$$V=\int_{-1}^1 2\pi r(x)h(x)\,dx$$

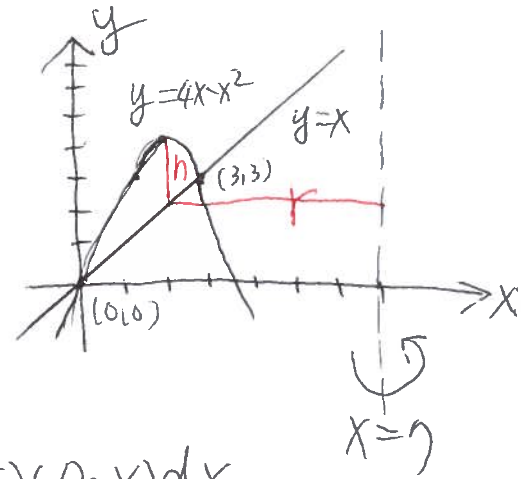
$$=2\pi\int_{-1}^1 (1-x)(2-2x^2)\,dx$$

$$=2\pi\int_{-1}^1 2-2x-2x^2+2x^3\,dx=2\pi\left[2x-x^2-\frac{2}{3}x^3+\frac{x^4}{2}\right]_{-1}^1$$

$$=2\pi\cdot\left[2\cdot 2-\frac{2}{3}\cdot 2\right]=\underline{\frac{16}{3}\pi}$$

§6.3

(22) Given $y=x$, $y=4x-x^2$, about $x=7$



intersection points: $x=4x-x^2$

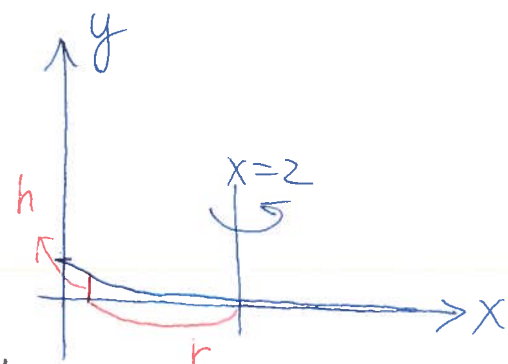
$\Rightarrow x^2-3x=0$, $x=0$ or $3 \Rightarrow (0,0)$, $(3,3)$

$h=4x-x^2-x$, $r=7-x$, $0\le x\le 3$

$$V=\int_0^3 2\pi\, r(x)h(x)\,dx=\int_0^3 2\pi(3x-x^2)(7-x)\,dx.$$

(24) Given $y=\frac{1}{1+x^2}$, $y=0$, $x=0$, $x=2$

about $x=2$

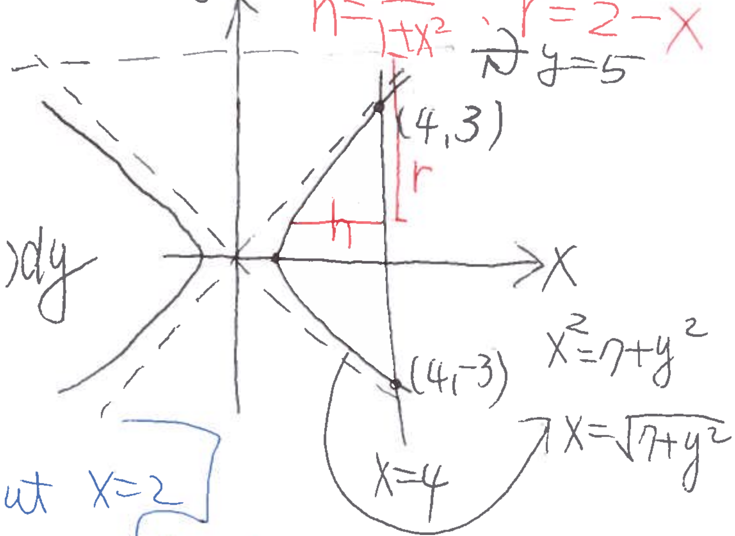


$$V=\int_0^2 2\pi\frac{1}{1+x^2}\cdot(2-x)\,dx.$$

$h=\frac{1}{1+x^2}$, $r=2-x$

(26) Given $x^2-y^2=7$, $x=4$,

about $y=5$.



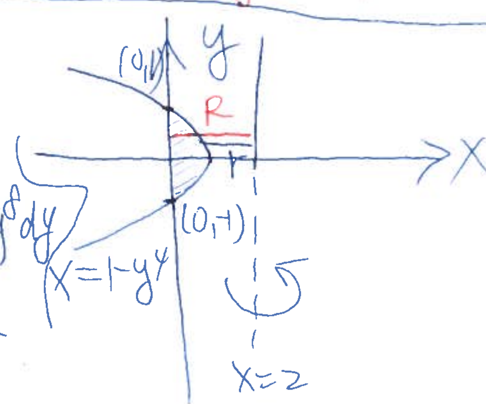
$$V=\int_{-3}^{3}2\pi\left(4-\sqrt{7+y^2}\right)(5-y)\,dy$$

$h=4-\sqrt{7+y^2}$, $r=5-y$

(40) Given $x=1-y^4$, $x=0$, about $x=2$

By Cross-Section (which is easier)

$R=2$. $r=2-(1-y^4)=1+y^4$.

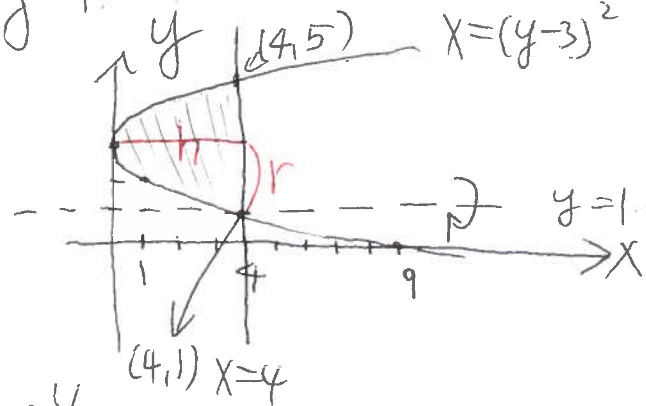


$$V=\pi\int_{-1}^{1}4-(1+y^4)^2\,dy=2\pi\int_0^1 3-2y^4-y^8\,dy$$

$$=2\pi\left[3y-\frac{2}{5}y^5-\frac{y^9}{9}\right]_0^1=2\pi\,\frac{112}{45}=\frac{224}{45}\pi$$

(42) Given $x=(y-3)^2$, $x=4$, about $y=1$

$h=4-(y-3)^2$, $r=y-1$



$$V=\int_1^5 2\pi\,(y-1)\left[4-(y-3)^2\right]dy$$

$$=2\pi\int_0^4 4-(u-2)^2\,du = 2\pi\int_0^4 -u^2+4u\,du$$

let $u=y-1$

$du=dy$

$$=2\pi\cdot\left[\frac{-u^3}{3}+2u^2\right]_0^4=2\pi\left[\frac{-4^3}{3}+2\cdot 4^2\right]$$

$$=2\pi\cdot\frac{32}{3}=\frac{64}{3}\pi.$$