Honor Calculus, Math 1450 − HW5 − Solution

(1) Math Induction:

To prove P(n) is right for all integer n,

<u>first</u>, we check, as n=1, P(1) is right.

<u>Second</u>, assume as n=k, P(k) is right, then we use P(k) to prove p(k+1) is right.

<u>Thus</u>, We can say P(n) is right for all integer n.

(i) $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

As n=1, we have $\sum_{j=1}^{1} j = 1 = \frac{1(1+1)}{2} = RHS.$

Assume, as n=k, we have $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$. Then, as n=k+1

$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)\left[\frac{k}{2}+1\right]$$

$$= (k+1)\cdot\frac{k+2}{2} = \frac{(k+1)[(k+1)+1]}{2} = RHS.$$

which means, as n=k+1, the formula is right.

Thus, by Math induction, $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ for all integer n.

(ii) $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

As n=1, we have $\sum_{j=1}^{1} j^2 = 1 = \frac{1(1+1)(2\cdot1+1)}{6} = RHS.$

Assume, as n=k, we have $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, Then,

as n=k+1, $\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$=(k+1)\cdot\left[\frac{k(2k+1)}{6} + (k+1)\right] = (k+1)\left[\frac{2k^2+k}{6} + \frac{6k+6}{6}\right]$$

=(see P.2)

1.

$$= (k+1)\left[\frac{2k^2+7k+6}{6}\right] = (k+1)\left[\frac{(k+2)(2k+3)}{6}\right]$$

$$= \frac{(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right]}{6} = RHS$$

which means, as $n=k+1$, the formula is right.

Thus, by math induction, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ for all integer n

(2) ① To show $\int_0^a x\,dx = \frac{a^2}{2}$ by Riemann sums.

Let $P = \{0, \frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \frac{4a}{n}, \ldots \frac{na}{n}\}$ be a partition of interval $[0,a]$

By def. of Riemann Sum, we have the Riemann Sum of $\int_0^a x\,dx$

is $$S_R = \frac{a}{n}\cdot\left[\frac{a}{n} + \frac{2a}{n} + \frac{3a}{n} + \cdots + \frac{na}{n}\right]$$

$$= \frac{a}{n}\cdot\frac{a}{n}\cdot\left[1+2+3+\cdots+n\right] = \frac{a^2}{n^2}\cdot\frac{n(n+1)}{2}$$ (by Question (1))

and $$\int_0^a x\,dx = \lim_{n\to\infty} S_R = \lim_{n\to\infty}\frac{a^2 n(n+1)}{2n^2} \underset{\uparrow}{=} \frac{a^2}{2}$$

leading coefficient.

② To show $\int_0^a x^2\,dx = \frac{a^3}{3}$ by Riemann Sums,

Using the same partition we had above. We have the Riemann Sum of $\int_0^a x^2\,dx$ is

$$S_R = \frac{a}{n}\left[\left(\frac{a}{n}\right)^2 + \left(\frac{2a}{n}\right)^2 + \left(\frac{3a}{n}\right)^2 + \cdots + \left(\frac{na}{n}\right)^2\right]$$

$$= \frac{a}{n}\left(\frac{a}{n}\right)^2\left[1^2+2^2+3^2+\cdots+n^2\right] = \frac{a^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$ (by Question (1))

and $$\int_0^a x^2\,dx = \lim_{n\to\infty} S_R = \lim_{n\to\infty}\frac{a^3\cdot n(n+1)(2n+1)}{6n^3} = \frac{2a^3}{6} = \frac{a^3}{3}$$

(3) Section 5.2

52. To show $\int_0^1 \sqrt{1+x^2}\,dx \le \int_0^1 \sqrt{1+x}\,dx$

By Comparison Properties of the integral

Since $x^2 \le x$ for $0 \le x \le 1$, then $1+x^2 \le 1+x$ implies $\sqrt{1+x^2} \le \sqrt{1+x}$ for $0 \le x \le 1$. So we have

$$\int_0^1 \sqrt{1+x^2}\,dx \le \int_0^1 \sqrt{1+x}\,dx$$

54. To show $\frac{\sqrt{2}}{24}\pi \le \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x\,dx \le \frac{\sqrt{3}\pi}{24}$.

By Comparison Properties of the integral

Since $\frac{\sqrt{2}}{2} = \cos\frac{\pi}{4} \le \cos x \le \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ as $\frac{\pi}{6} \le x \le \frac{\pi}{4}$.

Then we have

$$\underbrace{\frac{\sqrt{2}}{2}\cdot\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}_{\frac{\sqrt{2}\pi}{24}} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x\,dx \le \underbrace{\frac{\sqrt{3}}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}_{\frac{\sqrt{3}\pi}{24}}$$

$$\Rightarrow \quad \frac{\sqrt{2}\pi}{24} \qquad\qquad \frac{\sqrt{3}\pi}{24}$$

(4) Section 5.3

24. $\int_1^{\overset{8}{\cancel{6}}} \sqrt[3]{x}\,dx = \int_1^{\overset{8}{\cancel{6}}} x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}}\Big|_1^{\overset{8}{\cancel{6}}} = \frac{3}{4}\left[\overset{8}{\cancel{6}}^{\frac{4}{3}} - 1^{\frac{4}{3}}\right]$

$= \frac{3}{4}\left(\overset{16-1}{6\sqrt[3]{6}-1}\right) = \frac{45}{4}.$

28. $\int_0^1 (3+x\sqrt{x})\,dx = \int_0^1 3 + x^{\frac{3}{2}}\,dx = \left[3x + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1$

$= 3(1-0) + \frac{2}{5}\left(1^{\frac{5}{2}} - 0^{\frac{5}{2}}\right) = 3 + \frac{2}{5} = \frac{17}{5}$

Section 5.3

36 $\int_0^1 10^x dx = \frac{1}{\ln 10}\left[10^x\right]_0^1 = \frac{1}{\ln 10}\left[10^1 - 10^0\right] = \frac{9}{\ln 10}$

(what is $(10^x)'$? $(10^x)' = (e^{\ln 10^x})' = (e^{x \ln 10})' = \ln 10 \cdot 10^x$)

40. $\int_1^2 \frac{4+u^2}{u^3} du = \int_1^2 \left(\frac{4}{u^3} + \frac{1}{u}\right) du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2$

$= -2(2^{-2} - 1^{-2}) + \ln|2| - \ln|1|$

$= -2\left(\frac{1}{4} - 1\right) + \ln 2 - 0 = \frac{3}{2} + \ln 2$

54. Given $g(x) = \int_{\tan(x)}^{x^3} \frac{1}{\sqrt{2+t^4}} dt$, then, by Fundamental thm of Calculue we have, let a be a constant such that

$$g(x) = \int_a^{x^3} \frac{1}{\sqrt{2+t^4}} dt - \int_a^{\tan(x)} \frac{1}{\sqrt{2+t^4}} dt.$$

Then $g'(x) = \frac{1}{\sqrt{2+(x^3)^4}} \cdot 3x^2 - \frac{1}{\sqrt{2+(\tan(x))^4}} \cdot \sec^2(x).$

56. Given $y = \int_{\cos(x)}^{5x} \cos(u^2) du$, Then, by FTC, we have, let a be a constant such that

$$y = \int_a^{5x} \cos(u^2) du - \int_0^{\cos(x)} \cos(u^2) du.$$

Then $y' = 5\cos((5x)^2) + \sin(x) \cdot \cos((\cos(x))^2)$

Section 5.3

66. $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \dots + \sqrt{\frac{n}{n}}\right)$

$\uparrow$ length of each partition

$\leftarrow$ the function is $\sqrt{x}$ from 0 to 1.

$$= \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}\left[1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}.$$

68. If f is continuous and g, h are differentiable. Then

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = \frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt - \int_a^{g(x)} f(t)\,dt\right]$$

$$= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x).$$

70. (a) To show $\cos(x^2) \geq \cos(x)$ for $0 \leq x \leq 1$.

Since cosine function is a decreasing function on $[0,1]$ and $x^2 \leqq x$ if $x \in [0,1]$, Then

$$\cos(x^2) \geq \cos(x).$$

(b) To Deduce that $\int_0^{\frac{\pi}{6}} \cos(x^2)\,dx \geq \frac{1}{2}$. We have

Since $(0,\frac{\pi}{6}) \subset [0,1]$. Then $\cos(x^2) \geq \cos(x)$ on $(0,\frac{\pi}{6})$

and $\int_0^{\frac{\pi}{6}} \cos(x^2)\,dx \geq \int_0^{\frac{\pi}{6}} \cos(x)\,dx = \sin(x)\Big|_0^{\frac{\pi}{6}}$

$$= \sin\frac{\pi}{6} - \sin 0 = \frac{1}{2}$$

(5)

(i) $\int e^x \sin(e^x)dx = \int \sin(u)du = -\cos(u)+C$

$= -\cos(e^x)+C$

$\boxed{\text{Let } u=e^x,\ du=e^x dx}$

(ii) $\int \frac{\log x}{x}dx = \int u\,du = \frac{u^2}{2}+C_1 = \frac{(\log(x))^2}{2}+C$

$\boxed{\text{Let } u=\log(x),\ du=\frac{dx}{x}}$

(iii) $\int \frac{x}{\sqrt{1-x^4}}dx = \int \frac{1}{2}\,\frac{\cos(u)du}{\sqrt{1-\sin^2(u)}} = \int \frac{1}{2}\,\frac{\cos(u)\,du}{\cos(u)} = \int \frac{1}{2}du$

$\boxed{\text{Let } x^2=\sin(u),\ 2xdx=\cos(u)du}$ $\boxed{1-\sin^2(u)=\cos^2(u)}$

$= \frac{u}{2}+C = \frac{1}{2}\arcsin(x^2)+C$

$\boxed{x^2=\sin(u) \Rightarrow u=\arcsin(x^2)}$

(iv) $\int x^2\sin(x)dx = -x^2\cos(x)+2x\cdot\sin(x)+2\cos(x)+C$

u	dV	sign
x^2	$\sin(x)$	+
$2x$	$-\cos(x)$	−
2	$-\sin(x)$	+
0	$\cos(x)$	−
		+

(v) $\int x\sqrt{1-x^2}dx = \int -\frac{\sqrt{u}}{2}du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}+C_1 = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$

$\boxed{\text{Let } u=1-x^2,\ du=-2xdx}$

(5)

(vi) $\int (\log(x))^2 dx \overset{\uparrow}{=} x(\log(x))^2 - \int 2\log(x) dx$

Let $u = (\log(x))^2$ $dv = dx$

$du = 2\cdot\log(x)\cdot\frac{dx}{x}$ $v = x$

Let $u = \log(x)$ $dv = dx$

$du = \frac{1}{x}dx$ $v = x$

$= x(\log(x))^2 - 2x\log(x) + 2\int dx$

$= x(\log(x))^2 - 2x\log(x) + 2x + C$

(vii) $\int \sqrt{x}\log(x)dx \overset{\uparrow}{=} \frac{2}{3}x^{\frac{3}{2}}\log(x) - \int \frac{2}{3}x^{\frac{1}{2}}dx$

Let $u = \log(x)$ $dv = \sqrt{x}dx$

$du = \frac{dx}{x}$ $v = \frac{2}{3}x^{\frac{3}{2}}$

$= \frac{2}{3}x^{\frac{3}{2}}\log(x) - \frac{4}{9}x^{\frac{3}{2}} + C$

(viii) $\int \frac{dx}{x\log(x)} \overset{\uparrow}{=} \int \frac{du}{u} = \ln|u| + c$

$= \ln|\log(x)| + c$

Let $u = \log(x)$, $du = \frac{dx}{x}$

(6)

Section 7.1

18. $\int e^{-\theta}\cos 2\theta\, d\theta \overset{\uparrow}{=} -e^{-\theta}\cos(2\theta) - 2\int e^{-\theta}\sin(2\theta)d\theta$

Let $u = \cos 2\theta$ $dv = e^{-\theta}d\theta$

$du = -2\sin 2\theta$ $v = -e^{-\theta}$

$= -e^{-\theta}\cos(2\theta) - 2\left[-e^{-\theta}\sin(2\theta) + \int 2\cos(2\theta)\cdot e^{-\theta}d\theta\right]$

Let $u = \sin(2\theta)$ $dv = e^{-\theta}d\theta$

$du = 2\cos(2\theta)$ $v = -e^{\theta}$

$\Rightarrow$ See NexT Page

$$\Rightarrow \int e^{-\theta}\cos(2\theta)d\theta = -e^{-\theta}\cos(2\theta) + 2e^{-\theta}\sin(2\theta) - 4\int e^{-\theta}\cos(2\theta)d\theta + c$$

$$\Rightarrow 5\int e^{-\theta}\cos(2\theta)d\theta = -e^{-\theta}\cos(2\theta) + 2e^{-\theta}\sin(2\theta) + c$$

$$\Rightarrow \int e^{-\theta}\cos(2\theta)d\theta = \frac{-e^{-\theta}\cos(2\theta) + 2e^{-\theta}\sin(2\theta)}{5} + c$$

28. $\int_1^2 \frac{(\ln x)^2}{x^3}dx.$ $\Rightarrow$ Consider $\int \frac{(\ln x)^2}{x^3}dx = -\frac{(\ln x)^2}{2x^2} + \int \frac{(\ln x)}{x^3}dx$

$$\boxed{\begin{array}{l}\text{Let } u=(\ln x)^2 \quad dv=\frac{dx}{x^3} \\ du=2(\ln x)\frac{dx}{x} \quad v=\frac{-1}{2}\frac{1}{x^2}\end{array}}$$

$$\boxed{\begin{array}{l}\text{Let } u=(\ln x) \leftarrow dv=\frac{dx}{x^3} \\ du=\frac{dx}{x} \leftarrow v=\frac{-1}{2}\frac{1}{x^2}\end{array}}$$

$$= -\frac{(\ln x)^2}{2x^2} - \frac{(\ln x)}{2x^2} + \frac{1}{2}\int\frac{dx}{x^3} = -\frac{(\ln x)^2}{2x^2} - \frac{(\ln x)}{2x^2} - \frac{1}{4x^2} + C$$

Then $\int_1^2 \frac{(\ln x)^2}{x^3}dx = \left[-\frac{(\ln x)^2}{2x^2} - \frac{(\ln x)}{2x^2} - \frac{1}{4x^2}\right]_1^2 = -\left(\frac{(\ln 2)^2}{8} + \frac{\ln 2}{8} + \frac{1}{16}\right)$

$$+\left(\cancelto{0}{\frac{\ln 1^2}{2}} + \cancelto{0}{\frac{\ln 1}{2}} + \frac{1}{4}\right) = \frac{3}{16} - \frac{(\ln 2)^2}{8} - \frac{\ln 2}{8}.$$

32. $\int_0^t e^s \sin(t-s)ds$

First we deal with $\int e^s \sin(t-s)ds$, we have .

$$\int e^s \sin(t-s)ds = e^s\sin(t-s) + \int e^s\cos(t-s)ds$$

$$\boxed{\begin{array}{l}\text{Let } u=\sin(t-s) \quad dv=e^s ds \\ du=\cos(t-s)ds \quad v=e^s\end{array}}$$

$$= e^s\sin(t-s) + e^s\cos(t-s) - \int e^s\sin(t-s)ds$$

Let $u=\cos(t-s)$ $dv=e^s ds$

$du=-(-\sin(t-s))ds$ $v=e^s$

Then,

$$2\int e^s \sin(t-s)\,ds = e^s\sin(t-s) + e^s\cos(t-s)$$

$$\Rightarrow \int_0^t e^s \sin(t-s)\,ds = \frac{1}{2}e^s\sin(t-s) + \frac{1}{2}e^s\cos(t-s)\Big|_0^t$$

$$= \frac{1}{2}(e^t \cancelto{0}{\sin 0}) - \frac{1}{2}(e^0\sin(t)) + \frac{1}{2}e^t\cos(0) - \frac{1}{2}(e^0\cos(t))$$

$$= \frac{1}{2}e^t - \frac{e^0}{2}(\sin(t)+\cos(t))$$

36. $\int_0^\pi e^{\cos(t)}\sin(2t)\,dt = \int_0^\pi e^{\cos(t)}\, 2\sin(t)\cos(t)\,dt$

$$\boxed{\begin{array}{lll} \text{Let} & u = 2\cos(t) & dv = \sin(t)e^{\cos(t)}dt \\ & du = -2\sin(t)dt & v = -e^{\cos(t)} \end{array}}$$

$$= -2e^{\cos(t)}\cos(t)\Big|_0^\pi - 2\int_0^\pi \sin(t)e^{\cos(t)}\,dt$$

$$= -2e^{\cos(\pi)}\cos(\pi) + 2e^{\cos(0)}\cos(0) + 2\left[e^{\cos(t)}\right]_0^\pi$$

$$= -2e^{-1}\cdot(-1) + 2e + 2\left[e^{\cos(\pi)} - e^{\cos(0)}\right] = 4e^{-1}$$

48. $\int x^n e^x\,dx \overset{\uparrow}{=} x^n e^x - n\int x^{n-1}e^x\,dx$

$$\boxed{\begin{array}{lll} \text{Let} & u = x^n & dv = e^x dx \\ & du = (n)x^{n-1}dx & v = e^x \end{array}}$$