Honor Calculus, Math 1450 — Homework 4 Solution.

(1) Mean Value Theorem (MVT)

f is continuous on $[a,b]$, f is differentiable on (a,b)

Then there is a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(a) Assume f is differentiable on $\mathbb{R}$, and has two roots

Let a, b be two roots, we have $f(a)=0$, $f(b)=0$.

W.L.O.G, we assume $a<b$. By (MVT), we have

a $c\in(a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$$

$\Rightarrow f'(x)$ has at least one root which is c

(b) Assume f is twice differentiable on $\mathbb{R}$, and has three roots

Let a, b, d be three roots, $a<b<d$, we have $f(a)=0$, $f(b)=0$, $f(d)=0$

Using the conclusion of part (a), by MVT, we have

$c\in(a,b)$, $e\in(b,d)$ s.t $f'(c)=0$, $f'(e)=0$.

Since $f'(c)=0$, $f'(e)=0$ and f' is differentiable (since f'' exists)

By MVT, we have a $h\in(c,e)$ s.t

$$(f')'(h) = \frac{f'(e)-f'(c)}{e-c} = \frac{0-0}{e-c} = 0 \Rightarrow f''(h)=0$$

which means f'' has at least one root "h".

(2) Assume $f'(x) > g'(x)$ on (a,b) and $f(a) = g(a)$

By Mean value Theorem, let $F(x) = f(x) - g(x)$

Since both f, g are differentiable on (a,b), so is $F(x)$.

Since $f(a) = g(a) \Rightarrow F(a) = f(a) - g(a) = 0$.

let $y \in (a,b)$, there is a number $c \in (a,y)$ such that

$$F'(c) = \frac{F(y) - F(a)}{y-a} = \frac{F(y)}{y-a} \quad \text{— } (*)$$

Since $F'(x) = f'(x) - g'(x)$ and $f'(x) > g'(x)$ on (a,b).

$\Rightarrow F'(x) > 0$ on (a,b).

Then $(*)$ Implies $F(y) > 0$ for an arbitrary $y \in (a,b)$

$\Rightarrow F(y) > 0$ for all $y \in (a,b)$

$\Rightarrow f(y) > g(y)$ for all $y \in (a,b)$.

(3) let $g(x) = \sqrt{x+1}$, $f(x) = 1 + \frac{x}{2}$ for $x > 0$,

then $g'(x) = \frac{1}{2\sqrt{x+1}}$, $f'(x) = \frac{1}{2}$

We get $f'(x) > g'(x)$ for all $x > 0$ since $\frac{1}{\sqrt{x+1}} < 1$ for all $x > 0$

by Ex(2), we have

$$1 + \frac{x}{2} = f(x) > g(x) = \sqrt{x+1}.$$

(4) Given

43 (14) $f(x) = \cos^2 x - 2\sin x \,,\; 0 \le x \le 2\pi$

(a) Increasing and decreasing interval:

$$f'(x) = -2\cos(x)\cdot\sin(x) - 2\cos(x) = 0 \Rightarrow 2\cos(x)[\sin(x)-1] = 0$$

$$\Rightarrow \cos(x) = 0 \text{ or } \sin(x) = 1 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2},$$

Increasing interval: $[\frac{\pi}{2}, \frac{3\pi}{2}]$

decreasing intervals $[0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$

	0 to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to 2π
	$-$	$+$	$-$
$\cos(x)$	$+$	$-$	$+$
$\sin(x)-1$	$-$	$-$	$-$
	decrea.	increa.	decrea.

(b) $f(\frac{3\pi}{2}) = \cos^2(\frac{3\pi}{2}) - 2\sin(\frac{3\pi}{2}) = 1 \Leftarrow$ local max

$f(\frac{\pi}{2}) = \cos^2(\frac{\pi}{2}) - 2\sin(\frac{\pi}{2}) = -2 \Leftarrow$ local min

local min. local max.

(c) $f''(x) = 2\sin^2(x) - 2\cos^2(x) + 2\sin(x) = 0$

$$\Rightarrow 2\sin^2(x) - 2(1-\sin^2(x)) + 2\sin(x) = 0$$

$$\Rightarrow 4\sin^2(x) + 2\sin(x) - 2 = 0$$

$$\Rightarrow (\sin(x)+1)(4\sin(x)-2) = 0 \Rightarrow \sin(x) = -1,\; \sin(x) = \frac{1}{2}$$

$$\Rightarrow x = \frac{3\pi}{2} \text{ or } \frac{\pi}{6}, \frac{5\pi}{6}.$$

	0 to $\frac{\pi}{6}$	$\frac{\pi}{6}$ to $\frac{5\pi}{6}$	$\frac{5\pi}{6}$ to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to 2π
	$-$	$+$	$-$	$-$
$\sin(x)+1$	$+$	$+$	$+$	$+$
$4\sin(x)-2$	$-$	$+$	$-$	$-$

$\Rightarrow$ Concave up $(\frac{\pi}{6}, \frac{5\pi}{6})$

Concave down: $(0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$.

Inflection points: $(\frac{\pi}{6}, f(\frac{\pi}{6})),\; (\frac{5\pi}{6}, f(\frac{5\pi}{6})),\; (\frac{3\pi}{2}, f(\frac{3\pi}{2}))$

$= (\frac{\pi}{6}, -\frac{1}{4}) \quad = (\frac{5\pi}{6}, -\frac{1}{4}) \quad = (\frac{3\pi}{2}, 2)$

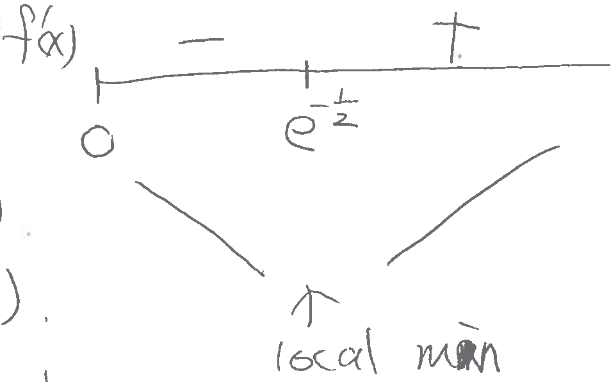
(4) Given

4.3 (16) $f(x) = x^2 \ln x$ on ~~ℝ~~ $x > 0$ (why?) (think about it!)

(a) $f'(x) = 2x\ln x + x = 0$

$\Rightarrow x(2\ln x + 1) = 0 \Rightarrow x = 0$ and $\ln x = -\frac{1}{2} \Rightarrow x = 0$ or $e^{-\frac{1}{2}}$

~~($2\ln x + 1 \neq 0$ since ln~~

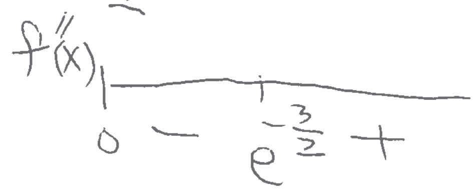


Increasing interval: $(e^{-\frac{1}{2}}, \infty)$

decreasing interval: $(0, e^{-\frac{1}{2}})$.

(b) $f(e^{-\frac{1}{2}}) = (e^{-\frac{1}{2}})^2 \ln e^{-\frac{1}{2}} = e^{-1} \cdot (-\frac{1}{2}) = -\frac{1}{2e}$ local min.

(c) $f''(x) = 2\ln x + 3 = 0 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-\frac{3}{2}}$



Concave up: $(e^{-\frac{3}{2}}, \infty)$

Concave down: $(0, e^{-\frac{3}{2}})$

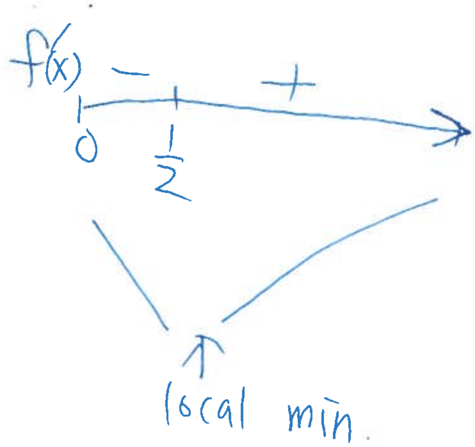
Inflection point: $(e^{-\frac{3}{2}}, f(e^{-\frac{3}{2}})) = (e^{-\frac{3}{2}}, -\frac{3}{2}e^{-3})$

4.3 (18) Given $f(x) = \sqrt{x}\, e^{-x}$, on ~~ℝ~~ $x > 0$

(a) $f'(x) = \frac{1}{2\sqrt{x}} e^{-x} - \sqrt{x} e^{-x} = 0$

$= \left(\frac{1 - 2x}{2\sqrt{x}}\right) e^{-x} = 0 \Rightarrow x = \frac{1}{2}$.

$f'(x) = \infty$ (DNE) $\Rightarrow x = 0$.



Increasing interval: $(\frac{1}{2}, \infty)$.

decreasing interval: $(0, \frac{1}{2})$.

(b) $f(\frac{1}{2}) = \sqrt{\frac{1}{2}}\, e^{-\frac{1}{2}} \leftarrow$ local min.

4.3 (18)

(c) $f''(x) = \left(\dfrac{-4\sqrt{x} - \frac{1}{\sqrt{x}}(1-2x)}{4x}\right)e^{-x} + \left(\dfrac{2x-1}{2\sqrt{x}}\right)e^{-x}$

$= \left(\dfrac{-4\sqrt{x} - \frac{1}{\sqrt{x}} \cancel{+2\sqrt{x}} + 4x\sqrt{x} \cancel{-2\sqrt{x}}}{4x}\right)e^{-x}$

$= \left(\dfrac{4x^2-4x-1}{4x\sqrt{x}}\right)e^{-x} = 0 \iff 4x^2-4x-1=0$

$\Rightarrow x = \dfrac{4 \pm 4\sqrt{2}}{8} = \dfrac{1\pm\sqrt{2}}{2}$ $\quad x>0 \Rightarrow x = \dfrac{1+\sqrt{2}}{2}$

$f''(x)$ DNE $\Rightarrow x=0$

f'': $-$ on $\left(0, \frac{1+\sqrt{2}}{2}\right)$, $+$ after $\frac{1+\sqrt{2}}{2}$

Concave up: $\left(\frac{1+\sqrt{2}}{2}, \infty\right)$

Concave down: $\left(0, \frac{1+\sqrt{2}}{2}\right)$ $\quad$ inflection point: $\left(\frac{1+\sqrt{2}}{2}, f\left(\frac{1+\sqrt{2}}{2}\right)\right)$

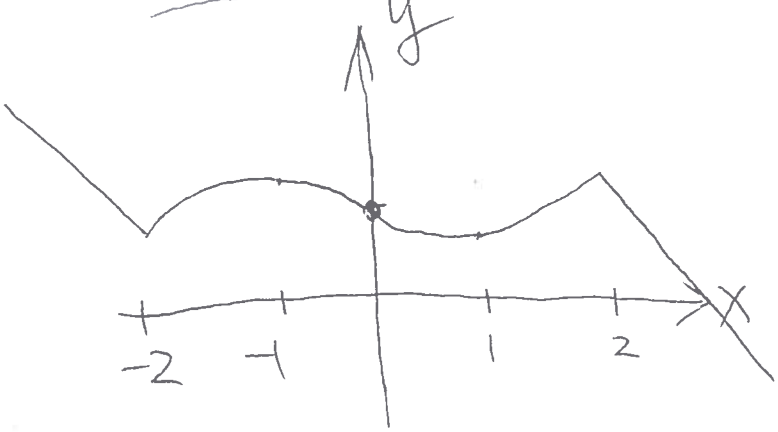
4.3 (26) Sketch the graph: $\to$ critical points.

$f'(1)=f'(-1)=0$, $f'(x)<0$ if $|x|<1$, $\Rightarrow f'(x)<0$, $-1<x<1$

$f'(x)>0$ if $1<|x|<2 \Rightarrow f'(x)>0$, $1<x<2$ $\quad -2<x<-1$,

$f'(x)=-1$ if $|x|>2 \Rightarrow f'(x)=-1$ $\quad x>2$ or $x<-2$

$f''(x)<0$ if $-2<x<0$. $\quad$ P.O.I: $(0,1)$



4.3

(68) Find a and b such that $f(x) = axe^{bx^2}$ have the maximum value $f(2)=1$.

Maximum value $f(2)=1 \Leftrightarrow f'(2)=0$. $\Rightarrow f'(x) = ae^{bx^2} + 2abx^2e^{bx^2}$

$f(2)=1 \Leftrightarrow \quad 1 = 2ae^{4b}$ —— (*)

$f'(2)=0 \Leftrightarrow \quad 0 = ae^{4b} + 8abe^{4b} \overset{e^{4b}>0}{\Longleftrightarrow} a+8ab=0 \Leftrightarrow a(1+8b)=0$

$\Rightarrow a=0$ or $b=-\frac{1}{8}$,

If $a=0$, put to (*), we get $1=0$, contradiction.

If $b=-\frac{1}{8}$ put to (*), we get $1=2ae^{-\frac{1}{2}} \Rightarrow a = \frac{e^{\frac{1}{2}}}{2}$.

Thus $a = \frac{\sqrt{e}}{2}$ and $b = -\frac{1}{8}$.

4.3

(72) Assume f and g are twice difference and $f''(x)\neq 0, g''(x)\neq 0$ $\forall x$.

Then

(a) If f and g are concave upward on I, we have

$f''(x)>0$ and $g''(x)>0$, for $x\in I$.

Then $(f+g)''(x) = f''(x) + g''(x) > 0$ for $x \in I$

$\Leftrightarrow$ $f+g$ is concave upward on I.

(b) If f is positive and concave upward on I, we have

$f''(x)>0$ for all $x\in I$. Let $g(x) = [f(x)]^2$

Then $g'(x) = 2[f(x)]\cdot f'(x)$ and

$g''(x) = 2[f'(x)]^2 + 2[f(x)]\cdot f''(x) > 0$

Since $[f'(x)]^2$ is always positive and $f(x)>0$ for all $x\in$

Thus, g is concave upward on I.

4.3 (76) (a) To show $e^x \geq 1+x$ for $x \geq 0$

By Ex.(2), we have if $f'(x) > g'(x)$ on (a,b) and $f(a) = g(a)$, then
$f(x) > g(x)$.

Let $f(x) = e^x$, $g(x) = 1+x$, as $x > 0$, we have $f(0) = 1 = g(0)$
and $f'(x) = e^x$, $g'(x) = 1$ implies $f'(x) > g'(x)$ for $x > 0$

$\Rightarrow e^x = f(x) > g(x) = 1+x$

For the "=" case, as $x=0$, we have $f(0) = g(0)$, so
$f(x) \geq g(x)$ as $x \geq 0$. #

(a') Another Method: Let $F(x) = e^x - (1+x)$, $x \geq 0$.

Since $F'(x) = e^x - 1 > 0$ as $x > 0$, $\Rightarrow F(x)$ is increasing as $x > 0$.

Since $F(0) = e^0 - (1+0) = 0$ and F is always increasing on $(0,\infty)$.

Then $F(x) > 0$ as $x > 0 \Leftrightarrow e^x > (1+x)$ as $x > 0$,

For "=" case, as $x=0$, we have $e^x = (1+x)$. Thus
$e^x \geq (1+x)$ as $x \geq 0$.

(b) Let $F(x) = e^x - (1 + x + \frac{1}{2}x^2)$, for $x \geq 0$.

Since $F'(x) = e^x - (1+x) > 0$ as $x > 0$ (by (a)) and $F(0) = 0$,

Then F is alway increasing and $F(0) = 0$ which implies
$e^x > 1 + x + \frac{x^2}{2}$ for $x > 0$,

For "=" case, Since, as $x=0$, $e^x = 1 + x + \frac{x^2}{2}$, Then
$e^x \geq 1 + x + \frac{x^2}{2}$, for $x \geq 0$.

(c). Conti.

4.3 (7b) Let $P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ where n is an integer.

(c) By (a), (b), we have $e^x \geq P_1(x)$, $e^x \geq P_2(x)$ as $x \geq 0$.

Then, by Mathematical induction,

assume $e^x \geq P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$, to show $e^x \geq P_{n+1}(x)$,

we have:

Let $F(x) = e^x - P_{n+1}(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}\right)$ $\forall x \geq 0$,

Since $F'(x) = e^x - \left(1 + \frac{x}{1!} + \cdots + \frac{nx^{n-1}}{n!} + \frac{(n+1)x^n}{(n+1)!}\right)$

$= e^x - \left(1 + x + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\right) = e^x - P_n(x) > 0$, $x > 0$

Then $F(x)$ is always increasing as $x > 0$ and $F(0) = 0$,

Thus $F(x) > 0$ for $x > 0$ $\Rightarrow$ $e^x > P_{n+1}(x)$, for $x > 0$.

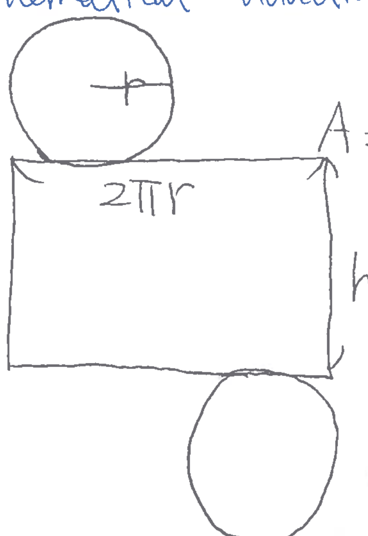
For "=" case, as $x = 0$, we have $e^x = P_{n+1}(x)$ $\Rightarrow$

$e^x \geq P_{n+1}(x)$ $\forall x \geq 0$.

Then, by Mathematical induction, $e^x \geq P_n(x)$, $x \geq 0$ for all integers n.

(5)

$\Rightarrow$



$A = \text{area} = \pi r^2 \times 2 + 2\pi r h$.

$= 2\pi r^2 + \frac{2\cancel{\pi} r}{\cancel{\pi} r^{\cancel{2}}}$

$= 2\pi r^2 + \frac{2}{r}$ $(r > 0)$

Volume $= 1$.

$\pi r^2 h$

implies $h = \frac{1}{\pi r^2}$

$\frac{dA}{dr} = 4\pi r - \frac{2}{r^2} = 0$

$\Rightarrow \frac{4\pi r^3 - 2}{r^2} = 0$

$\Rightarrow 4\pi r^3 - 2 = 0 \Rightarrow r = \sqrt[3]{\frac{1}{2\pi}}$

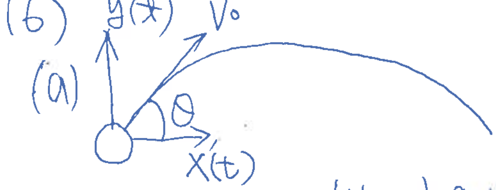
$\frac{dA}{dr}$: $-$ on $\left(0, \sqrt[3]{\frac{1}{2\pi}}\right)$, $+$ after $\sqrt[3]{\frac{1}{2\pi}}$

local min.

As $r = \sqrt[3]{\frac{1}{2\pi}}$, $h = \frac{1}{\pi r^2} = \frac{\sqrt[3]{4\pi^2}}{\pi}$,

this circular cylinder has the least surface area.

(6) (a)



Given $y(t) = -16t^2 + (v_0 \sin\theta)t$ and $x'(t) = v_0\cos\theta$

We have $x(t) = (v_0\cos\theta)t$. then $t = \frac{x}{v_0\cos\theta}$.

Thus, $y = -16\left(\frac{x}{v_0\cos\theta}\right)^2 + \frac{v_0\sin\theta}{v_0\cos\theta}x$ which is a parabola.

(b) Farthest before the projectile hitting the ground $\Rightarrow$ when? $y(t)=0$

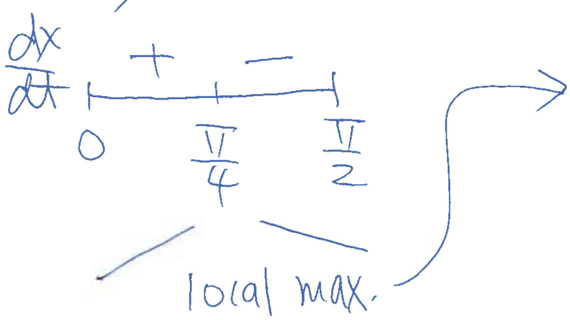
If $y(t)=0 \Rightarrow t(-16t + v_0\sin\theta) = 0 \Rightarrow t=0$ or $t = \frac{v_0\sin\theta}{16}$

Then $X(t) = (v_0\cos\theta)\cdot\frac{v_0\sin\theta}{16} = \frac{v_0^2}{16}\cos(\theta)\sin(\theta)$ ---- (*)

To find the maximum value of X, we have, for $0<\theta<\frac{\pi}{2}$

$$\frac{dx}{dt} = \frac{v_0^2}{16}\left[\cos^2(\theta) - \sin^2(\theta)\right] = 0 \Leftrightarrow \cos\theta + \sin\theta = 0 \text{ or } \cos\theta - \sin\theta = 0$$

$$\Rightarrow \cos\theta = -\sin\theta \text{ or } \cos\theta = \sin\theta \Rightarrow \theta = \cancel{\frac{3\pi}{4}} \text{ or } \frac{\pi}{4} \quad \left(0<\theta<\frac{\pi}{2}\right)$$



when $\theta = \frac{\pi}{4}$, X has the maximum value

$$X\left(\frac{\pi}{4}\right) = \frac{v_0^2}{16}\cdot\frac{1}{2} = \frac{v_0^2}{32}$$

by (*)

(7) (a) A motion of particle satifies $m\ddot{x} = -\frac{dV}{x}$ where m is mass of particle, x is position. $\dot{x}$ is velocity and $\ddot{x}$ is acceleration.

To prove $V(x) + \frac{1}{2}m\dot{x}^2 = $ constant, it is sufficiently to prove

$$\frac{d}{dt}\left(V + \frac{1}{2}m\ddot{x}^2\right) = 0$$

$$\frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt} = -m\ddot{x}\,\frac{dx}{dt} = -m\ddot{x}\dot{x}, \text{ and}$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{1}{2}m\,2\dot{x}\,(\dot{x})' = m\dot{x}\ddot{x} \Rightarrow$$

$$0=\frac{dV}{dt}+\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right)=\frac{d}{dt}\left(V+\frac{1}{2}m\dot{x}^2\right).$$

(b) Suppose the motion of a particle satisfies $m\ddot{x}=-kx$ where $m=1$, $k=2$ and $\dot{x}(0)=3$. Then, by (a), we have

$$-\frac{dV}{dx}=m\ddot{x}=-kx \quad\text{and}\quad V(x)+\frac{1}{2}m\dot{x}^2=\text{constant},$$

$$\Rightarrow \frac{dV}{dx}=kx \text{ implies } V=\frac{k}{2}x^2=x^2 \text{ and}$$

$$x^2+\frac{1}{2}m\dot{x}^2=\text{constant},\quad \forall t\geq 0.$$

Let $x(0)=0$, we have $x(0)+\frac{1}{2}\cdot 1\cdot[\dot{x}(0)]^2=0+\frac{9}{2}$.

where $\frac{9}{2}$ is ~~that~~ constant,

As $\dot{x}(t)=0$ we have the maximum distance, then

$$x^2(t)+\frac{1}{2}\cdot 1\cdot 0=\frac{9}{2} \Rightarrow x(t)=\frac{3}{\sqrt{2}}=\frac{3}{2}\sqrt{2}$$

(8).

4.4 (8) $\displaystyle\lim_{x\to 1}\frac{x^a-1}{x^b-1} \overset{(\frac{0}{0})}{\underset{L'}{=}} \lim_{x\to 1}\frac{a x^{a-1}}{b x^{b-1}}=\frac{a}{b}$.

(10) $\displaystyle\lim_{x\to 0}\frac{\sin 4x}{\tan(5x)}=\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\cdot\frac{5x}{\sin(5x)}\cdot\cos(5x)\cdot\frac{4}{5}\right)=1\cdot 1\cdot 1\cdot\frac{4}{5}=\frac{4}{5}$

(20) $\displaystyle\lim_{x\to 1}\frac{\ln(x)}{\sin\pi(x)} \overset{(\frac{0}{0})}{\underset{L'}{=}} \lim_{x\to 1}\frac{\frac{1}{x}}{\pi\cos(\pi x)}=\frac{1}{-\pi}=-\frac{1}{\pi}$.

(28) $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}\frac{2(\ln x)\cdot\frac{1}{x}}{1}=\lim_{x\to\infty}2\frac{\ln(x)}{x} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}2\frac{\frac{1}{x}}{1}=0$

4.4

(40) $\lim_{x\to-\infty} x^2 e^x \underset{(L')}{\overset{(0\cdot\infty)}{=}} \lim_{x\to-\infty} \frac{x^2}{e^{-x}} \underset{L'}{\overset{(\frac{\infty}{\infty})}{=}} \lim_{x\to-\infty} \frac{-2x}{-e^{-x}} \underset{L'}{\overset{(\frac{\infty}{\infty})}{=}} \lim_{x\to\infty} \frac{-2}{e^{-x}}$ DNE.

(42) $\lim_{x\to0^+} \sin(x)\cdot\ln(x) \overset{1\cdot(-\infty)}{=}$ DNE (NOT an indeterminate form of L'Hospital's)

(56) $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{\frac{x}{a}\cdot ab} = e^{ab}$