Honor Calculus, Math 1450 – Assignment 3 solution

§3.7 (10) Assume the velocity upward of a thrown ball is $80 (ft/s)$
(Free Fall)
and the height of the ball after t seconds is $S = 80t - 16t^2$

(a) Maximum height of ball $\Rightarrow$ the velocity of ball is 0.
For free fall problem, we have $a = 32 \ (ft/s^2)$.

$$d = v_i t + \frac{1}{2} a t^2, \quad V_f^2 = V_i^2 + 2ad, \quad V = V_i + at$$

So, we want to find d for $V_f = 0$, and $V_i = 80$

$$0 = 80^2 + 2(-32)d, \quad d = \frac{80^2}{64} = 100 (ft)$$

(b) For $V_i = 80$, if $S = 96$, we have $96 = 80t - 16t^2$

$$\Rightarrow 16t^2 - 80t + 96 = 0 \Rightarrow t^2 - 5t + 6 = 0 \Rightarrow (t-2)(t-3) = 0$$

$t = 2$ or 3; $\leftarrow$ on the way down

$\uparrow$

on the way up

as $t = 2$, $V = 80 + 2(-32) = \underline{16}$.

Similary, for the speed on its way down, it is $\underline{16}$.

§3.7 (22) In example 4, We have $A+B \to C$ and

$[A]=[B]=a$ moles/L, then $[C]=\dfrac{a^2kt}{(akt+1)}$, k is const.

(a) the rate of reaction is

$$\frac{d[C]}{dt}=\frac{a^2k(akt+1)-ak(a^2kt)}{(akt+1)^2}=\frac{a^2k}{(akt+1)^2}$$

(b) By (a), let $x=[C]$. we have

$$\frac{dx}{dt}=\frac{d[C]}{dt}=\frac{a^2k}{(akt+1)^2}$$ and $$k(a-x)^2=k(a-[C])^2$$

$$=k\left(a-\frac{a^2kt}{(akt+1)}\right)^2$$

$$=k\left(\frac{a(akt+1)-a^2kt}{(akt+1)}\right)^2=k\frac{a^2}{(akt+1)^2}$$

$$\Rightarrow \frac{dx}{dt}=k(a-x)^2.$$

(c) The concentration as $t\to\infty$ $\displaystyle\lim_{t\to\infty}[C]=\lim_{t\to\infty}\frac{a^2kt}{akt+1}=\frac{a^2k}{ak}=a$

(d) the rate of reaction as $t\to\infty$ is

$$\lim_{t\to\infty}\frac{d[C]}{dt}=\lim_{t\to\infty}\frac{a^2k}{(akt+1)^2}=0$$ (a, k are fixed).

(e) As $t\to\infty$, a moles per L A and a moles per L B will get a moles per L C and stop reaction.

§3.7 (34) Given $\frac{dP}{dt} = r_0 \left(1 - \frac{P(t)}{P_c}\right) P(t) - \beta P(t)$

(a pond and harvested population system)

(a) stable population $\Rightarrow \frac{dP}{dt} = 0$. (No changing w.r.t time)

(b) Given r_0 (birth rate) is 5%, β (harvesting rate) is 4%, P_c (sustain) is 10,000, we have

$$\frac{dP}{dt} = \frac{1}{20}\left(1 - \frac{P(t)}{10000}\right) P(t) - \frac{1}{25} P(t) = \frac{P(t)}{100} - \frac{(P(t))^2}{200000}$$

If $\frac{dP}{dt} = 0$, we have $0 = \frac{P}{100} - \frac{P^2}{200000}$

$\Rightarrow 2000P - P^2 = 0 \Rightarrow P(P - 2000) = 0$, $P = 2000$

(c) Given $\beta = 5\%$, we have $\frac{dP}{dt} = \frac{1}{20}\left(1 - \frac{P}{10000}\right) P - \frac{1}{20} P$

$$= -\frac{P^2}{20000}$$

$\Rightarrow P = 0$

§3.8 (4). Given a bacteria population $P(t)$, and $P(2) = 600$ (2 hrs), $P(8) = 75000$ (8 hrs). Since this culture grows with constant relative growth rate, We have $\frac{dP}{dt} = KP$ and $P(t) = P_0 e^{Kt}$ with initial population P_0

(a) $600 = P(2) = P_0 e^{K \cdot 2}$ —(1)

$75000 = P(8) = P_0 e^{K \cdot 8}$ —(2)

$\xRightarrow{\frac{(2)}{(1)}} \frac{75000}{600} = \frac{P_0 e^{8K}}{P_0 e^{2K}} = e^{6K}$

$\Rightarrow 125 = e^{6K} \Rightarrow \ln(125) = 6K \Rightarrow K = \frac{\ln(125)}{6}$

Then $600 = P_0 \cdot e^{\frac{\ln(125)}{6} \cdot 2} = P_0 e^{\frac{\ln(125)}{3}} = P_0 \cdot e^{\ln(125)^{\frac{1}{3}}} = P_0 (125)^{\frac{1}{3}}$

$= P_0 \cdot 5$

$\Rightarrow P_0 = 120.$

§3.8 (4)

(b) By (a), we have $P(t) = P_0 e^{kt} = 120 e^{\frac{\ln(125)}{6} \cdot t}$

(c) By (b), $P(5) = 120 e^{\frac{\ln(125)}{6} \cdot 5} = 120 e^{\ln(125)^{\frac{5}{6}}} = 120 \cdot (125)^{\frac{5}{6}}$

$= 120 \cdot 5^2 \sqrt{5} = 3000\sqrt{5}$

(d) $\frac{dP}{dt} = \frac{\ln(125)}{6} \cdot P(t)$. and as $t=5$, we have

$\left.\frac{dP}{dt}\right|_{t=5} = \frac{\ln(125)}{6} P(5) = \frac{\ln(125)}{6} 3000\sqrt{5} = [\ln(125)] \cdot 500\sqrt{5}$.

(e) Find t such that $P(t) = 200{,}000$, we have.

$200000 = 120 e^{\frac{\ln(125)}{6} t} = 120 (125)^{\frac{t}{6}}$

$\Rightarrow \frac{5000}{3} = (125)^{\frac{t}{6}} \Rightarrow \ln\left(\frac{5000}{3}\right) = \frac{t}{6} \ln(125)$

$\Rightarrow \frac{t}{6} = \frac{\ln\left(\frac{5000}{3}\right)}{\ln(125)} \Rightarrow t = 6 \frac{\ln\left(\frac{5000}{3}\right)}{\ln(125)}$.

§3.8 (18) Given \$1000 and interest 8%.

(a) After three years, we have (by $A_0(1+\frac{r}{n})^{nt}$)

(i) $1000(1.08)^3 =$ with annual compounding

(ii) $1000(1.02)^{12} =$ with quarterly "

(iii) $1000(1+\frac{8\%}{12})^{3 \cdot 12} = 1000(1.00666)^{36}$ with monthly "

(iv) $1000(1+\frac{8\%}{52})^{3 \cdot 52}$ with weekly "

(v) $1000(1+\frac{8\%}{360})^{3 \cdot 360}$ with daily "

(vi) $1000(1+\frac{8\%}{360 \cdot 24})^{3 \cdot 360 \cdot 24}$ with hourly "

(vii) $\lim_{n\to\infty} 1000(1+\frac{8\%}{n})^{3n} = 1000 \lim_{n\to\infty} (1+\frac{8\%}{n})^{\frac{n}{8\%} \cdot 0.24} = 1000 \cdot e^{0.24} \Leftarrow$ conti.

§3.8 (18)

(b) Give $0 \le t \le 3$.

after t years at 6% interest a \$1000 investment with continuously compounding will be.

$$A(t) = \lim_{n\to\infty} 1000\left(1+\frac{6\%}{n}\right)^{nt} = \lim_{n\to\infty} 1000\left(1+\frac{6\%}{n}\right)^{\frac{n}{6\%}\cdot(0.06t)}$$

$$= 1000 \cdot \lim_{n\to\infty}\left(1+\frac{6\%}{n}\right)^{\frac{n}{6\%}(0.06t)} = 1000\, e^{0.06t}.$$

Similarly, at 8% interest, we have

$$A(t) = 1000e^{0.08t}.$$

and at 10% interest, we have $A(t) = 1000e^{0.1t}$.

Graphs:

§3.9 (4)

Let the length of a rectangle be l, the width of a rectangle be w the area of this rectangle be R

we have $\frac{dl}{dt} = 8\ cm/s$, $\frac{dw}{dt} = 3\ cm/s$.

To Find the rate of change of area as $l=20$, $w=10$,

We have, $R = lw$ and

$$\frac{dR}{dt} = l\cdot\frac{dw}{dt} + \frac{dl}{dt}\cdot w$$ as $l=20$, $w=10$, we have

$$\left.\frac{dR}{dt}\right|_{\substack{l=20\\ w=10}} = 20\cdot 3 + 10\cdot 8 = 140\ cm/s$$

§3.9 (10) Given a trajectory of a particle is $y=\sqrt{1+x^3}$ and

$$\left.\frac{dy}{dt}\right|_{(x,y)=(2,3)} = 4\ cm/s$$, we have

$$\frac{dy}{dt} = \frac{d}{dt}\left((1+x^3)^{\frac{1}{2}}\right) = \frac{1}{2}\frac{1}{\sqrt{1+x^3}}\cdot 3x^2\cdot\frac{dx}{dt}$$

and $$4 = \left.\frac{dy}{dt}\right|_{(x,y)=(2,3)} = \frac{1}{2}\frac{1}{\sqrt{1+2^3}}\cdot 3\cdot 2^2\left.\frac{dx}{dt}\right|_{(x,y)=(2,3)}$$

$$\Rightarrow \left.\frac{dx}{dt}\right|_{(x,y)=(2,3)} = \frac{4}{3\cdot 2}\cdot\sqrt{9} = 2.$$

§3.9 (22) Given the trajectory of particle is $y=\sqrt{x}$,

and $\left.\frac{dx}{dt}\right|_{(x,y)=(4,2)} = 3\ cm/s$.

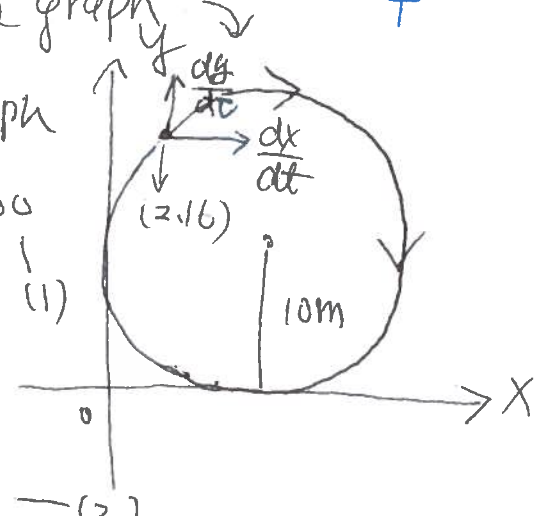
Then $\frac{dy}{dt} = \frac{1}{2}\frac{1}{\sqrt{x}}\cdot\frac{dx}{dt}$ and $\left.\frac{dy}{dt}\right|_{(x,y)=(4,2)} = \frac{1}{2}\cdot\frac{1}{\sqrt{4}}\cdot 3$

$= \frac{3}{4}$.

§3.9 (40) Given a Ferris wheel as the graph

the trajectory of ~~the~~ the given graph

will be $(x-10)^2+(y-10)^2 = 100$ — (1)



One revolution every 2 mins

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \frac{20\pi}{2\cdot 60} = \frac{\pi}{6}$ — (2)

As $y=16$, $x=2$ or 18 (we only need $x=2$)

Find $\frac{dy}{dt}$ as $x=2, y=16$.

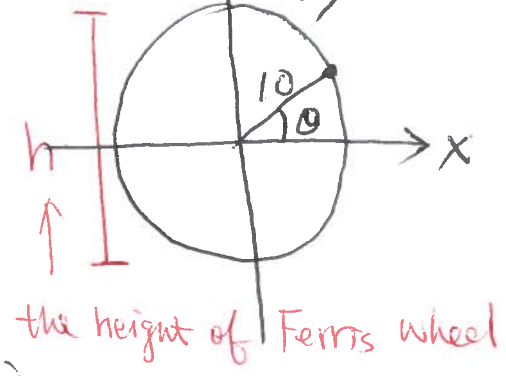
From (1), we have $2(x-10)\frac{dx}{dt}+2(y-10)\frac{dy}{dt}=0$

$\xRightarrow{x=2,\ y=16} -16\left.\frac{dx}{dt}\right|_{(2,16)} + 12\left.\frac{dy}{dt}\right|_{(2,16)} = 0 \Rightarrow \left.\frac{dx}{dt}\right|_{(2,16)} = \frac{3}{4}\left.\frac{dy}{dt}\right|_{(2,16)}$ — (3)

From (2) & (3), we have

$\sqrt{\left(\frac{3}{4}\left.\frac{dy}{dt}\right|_{(2,16)}\right)^2+\left(\left.\frac{dy}{dt}\right|_{(2,16)}\right)^2} = \frac{\pi}{6} \Rightarrow \left.\frac{dy}{dt}\right|_{(2,16)} = \frac{4}{5}\cdot\frac{\pi}{6} = \frac{2\pi}{15}\ (m/s)$

§ 3.9 (40) Another Method (by Meagan Woodford)



$$h - 10 = r \cdot \sin\theta$$
$$= 10\sin\theta$$

As $h = 16 \Rightarrow \sin\theta = \frac{6}{10} \Rightarrow \cos\theta = \frac{8}{10}$

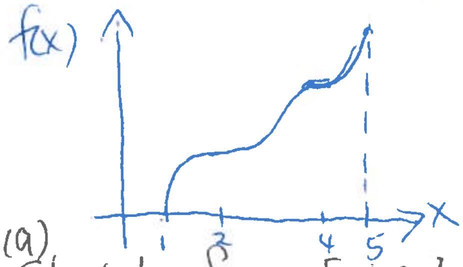
Assume one revolution every 2 mins

$$\Rightarrow \frac{d\theta}{dt} = \frac{2\pi}{2 \cdot 60} = \frac{\pi}{60}$$

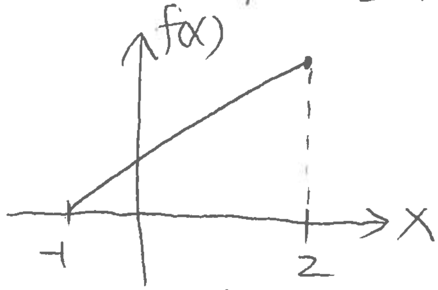
Then $\left.\frac{dh}{dt}\right|_{h=16} = 10 \cdot \cos\theta \cdot \left.\frac{d\theta}{dt}\right|_{\cos\theta = \frac{8}{10}} = \cancel{10} \cdot \frac{8}{\cancel{10}} \cdot \frac{\pi}{60} = \frac{8\pi}{60} = \frac{2\pi}{15} \left(m/s\right)$

§4.1 (10)

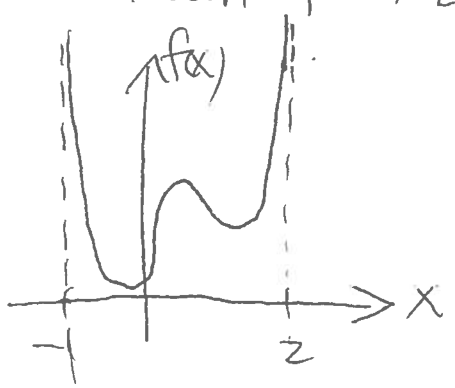
Sketh f which is continuous on [1,5] and has no local max. or min. but 2 and 4 are critical numbers



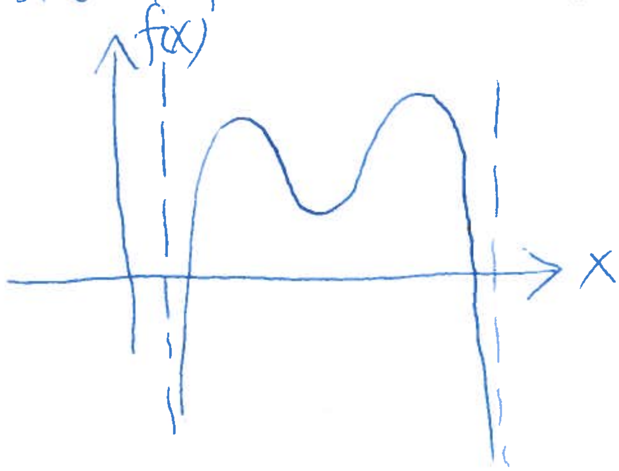
§4.1 (12) (a) Sketch f on [-1,2] which has an abs. max but no local max.



(b) Sketch f on [-1,2] which has a local max but no abs. max.

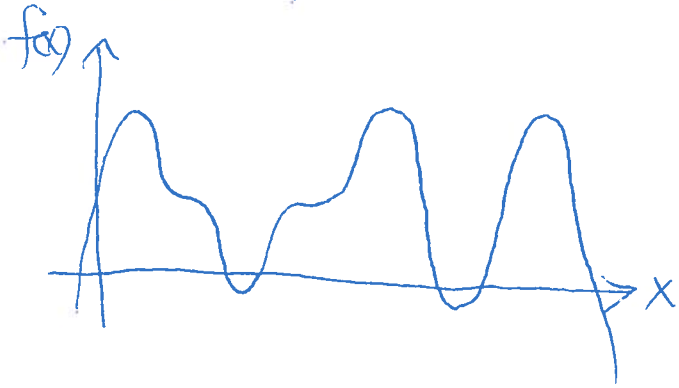


§4.1 (14) (a) Sketch f has two local max, one local min, but no abs. min.



§4.1 (14)(b)

Sketch f which has three local min, two local max, seven critical number



§4.1 (48) Given $f(x) = x^3 - 3x + 1$ on $[0,3]$.

To find abs. max and abs. min,

First, check the critical points $\Rightarrow f'(x) = 3x^2 - 3 = 0$

$\Rightarrow x = \pm 1$ (only $1 \in [0,3]$)

Then $f(1) = 1^3 - 3 + 1 = -1$

Second, check two endpoints: $f(0) = 0 - 3\cdot 0 + 1 = 1$ and

$f(3) = 3^3 - 3\cdot 3 + 1 = 19$

$\Rightarrow f(3) = 19$ is abs. max and $f(1) = -1$ is abs. min.

§4.1 (52) Given $f(x) = (x^2-1)^3$ on $[-1,2]$

To find abs. max and abs min, we have

first, check the critical point $\Rightarrow f'(x) = 3(x^2-1)^2 \cdot 2x = 0$

$\Rightarrow x = 0$ or 1 or -1

Then $f(0) = -1$, $f(1) = 0$, $f(-1) = 0$

Second, check the endpoints: $f(-1) = 0$, $f(2) = 3^3 = 27$

$\Rightarrow f(2) = 27$ is abs. max and $f(0) = -1$ is abs. min.

§4.1 (56) Given $f(t) = \sqrt[3]{t}\,(8-t)$ on $[0, 8]$.

First, check the critical point: ① $f'(t) = \frac{1}{3}\frac{1}{t^{\frac{2}{3}}}(8-t) + \sqrt[3]{t}(-1) = 0$

$\Rightarrow \frac{1}{t^{\frac{2}{3}}}\left[\frac{8-t}{3} - t\right] = 0 \Longrightarrow t = 2.$

② If $f'(t)$ DNE $\Rightarrow t = 0.$

$\Rightarrow f(0) = 8 \cdot 0 = 0.\quad f(2) = (\sqrt[3]{2})\,6$

Second, check the endpoints: $f(0) = 0.\ f(8) = 0.$

Then, $f(2) = 6\sqrt[3]{2}$ is abs. max and $f(0) = f(8) = 0$ are abs. min.

§4.1

(60) Given $f(x) = x - \ln x$ on $[\frac{1}{2}, 2]$.

First, check the critical point

$f'(x) = 1 - \frac{1}{x} \Rightarrow f'(x) = 0$ implies $x = 1$.

(we don't consider "$f'(x)$ DNE" since $0 \notin [\frac{1}{2}, 2]$)

$f(1) = 1 - \ln 1 = 1$

Second, check the endpoints $f(\frac{1}{2}) = \underbrace{\frac{1}{2} - \ln\frac{1}{2}}_{>1}$, $f(2) = \underbrace{2 - \ln 2}_{>1.}$ (since $\ln e > \ln 2$)

Then $f(2) = 2 - \ln 2$ is abs. max

and $f(1) = 1$ is abs. min.

§4.1

(62) Given $f(x)=e^{-x}-e^{-2x}$ on $[0,1]$

First, check the critical point: $f'(x)=-e^{-x}+2e^{-2x}=0$

$\Rightarrow 2e^{-2x}=e^{-x} \Rightarrow 2=e^{x} \Rightarrow x=\ln 2$.

$f(\ln 2)=e^{-\ln 2}-e^{-2\ln 2}=e^{\ln 2^{-1}}-e^{\ln 2^{-2}}=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$.

Second, check the endpoints: $f(0)=e^{0}-e^{0}=0$, $f(1)=e^{-1}-e^{-2}=\frac{e-1}{e^{2}}$

$\Rightarrow f(1)=\cancel{\frac{e-1}{e^{2}}}\ \frac{1}{4}$ is abs. max and $f(0)=0$ is abs. min. $=0.~~8646~~$ 0.2325

§4.1 (70)

$F(\theta)=\dfrac{\mu W}{\mu\sin\theta+\cos\theta}$ $\quad 0\le\theta\le\frac{\pi}{2}$

$$\frac{dF(\theta)}{d\theta}=\frac{-\mu W\cdot[\mu\cos\theta-\sin\theta]}{(\mu\sin\theta+\cos\theta)^{2}}=0 \Rightarrow$$ critical point θ satisfies $\mu\cos\theta-\sin\theta=0$

$$\Rightarrow \tan\theta=\mu \Rightarrow F(\theta)=\frac{\mu W}{\mu\frac{\mu}{\sqrt{\mu^{2}+1}}+\frac{1}{\sqrt{\mu^{2}+1}}}=\frac{(\mu W)(\sqrt{\mu^{2}+1})}{\mu^{2}+1}=\frac{\mu W}{\sqrt{\mu^{2}+1}}$$

$\sin\theta=\frac{\mu}{\sqrt{\mu^{2}+1}}$, $\cos\theta=\frac{1}{\sqrt{\mu^{2}+1}}$

For endpoints we have, $F(0)=\frac{\mu W}{\mu 0+1}=\mu W$, $F(1)=\frac{\mu W}{\mu+0}=W$.

as $\tan\theta=\mu$

$\Rightarrow F(\theta)$ has abs. min $\frac{\mu W}{\sqrt{\mu^{2}+1}}$. $\left(\sin\theta\ \frac{\mu}{\sqrt{\mu^{2}+1}}<1\right)$

§4.1 (76) Assume f has a local min. at c. if $g(x)=-f(x)$

Show that g has a local max at c.

f has a local min at $c \iff f'(c)=0$ and $f(c)\le f(x)$

since $g'(x)=-f'(x)$, where x is near c.

$\iff g'(0)=-f'(0)=0$ and $g(c)=-f(c)\ge -f(x)=g(x)$

where x is near c

$\iff g'(0)=0$ and $g(c)\ge g(x)$ where x is near c

$\iff g$ has a local max at c. (See another proofs at the end)

§4.1 (78) Given $f(x)=ax^3+bx^2+cx+d$, $a\ne 0$.

(a) Check critical number by $f'(x)=3ax^2+2bx+c=0$

If $f'(x)=0$ has two solution $\Rightarrow$ Two critical points

one " $\Rightarrow$ one "

no $\Rightarrow$ no critical point

By Quadratic Formula, the discriminant is

$4b^2-12ac$

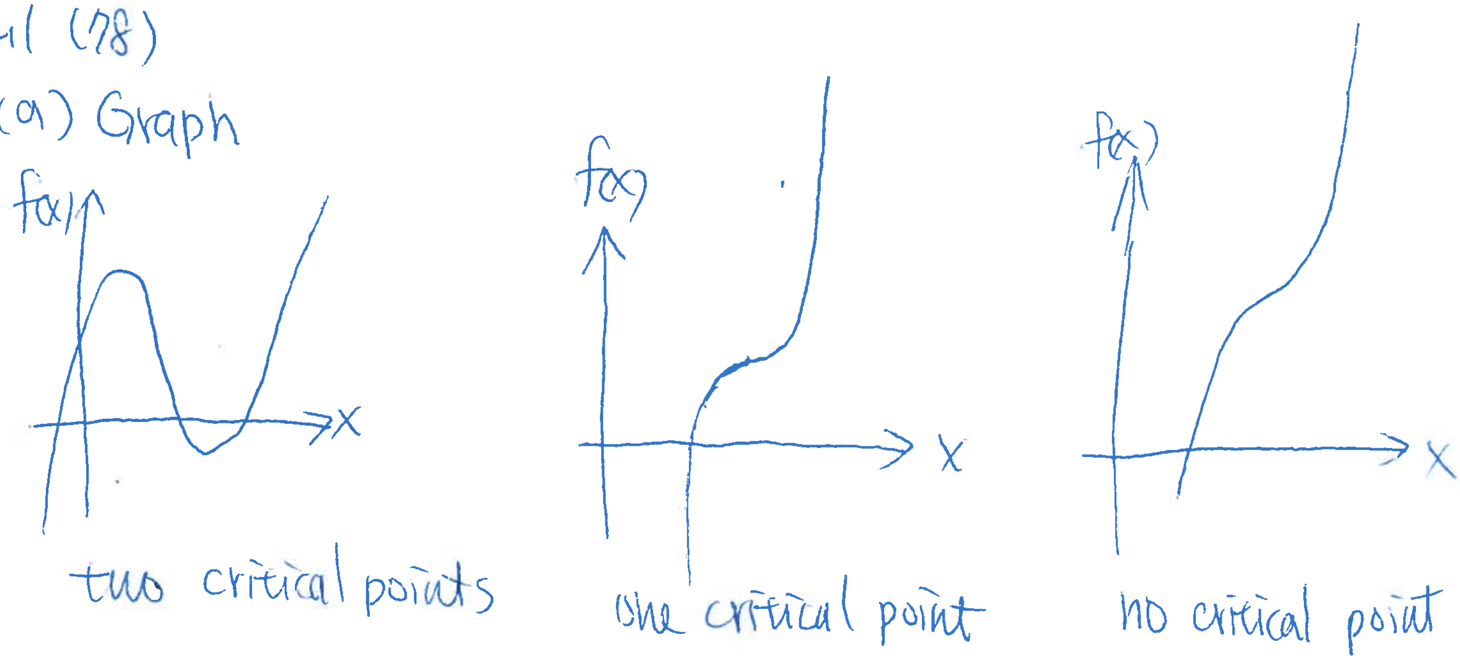
Then $4b^2-12ac>0 \Rightarrow$ Two critical points

$4b^2-12ac=0 \Rightarrow$ one critical point

$4b^2-12ac<0 \Rightarrow$ No critical point.

§4.1 (78)

(a) Graph



(b) For cubic function f,

We can have ① one local min and one local max

② No local extreme.

§4.1 (76) (by Neelesh Mutyala)

f has a local min. at $c \Leftrightarrow f(x) \geq f(c)$ where x is near c

$\Leftrightarrow g(x) = -f(x) \leq -f(c) = -g(x)$ where x is near c

$\Leftrightarrow g(x) \leq g(c) \Leftrightarrow g$ has a local max. at c.

§4.1 (76) (by Praneeth Kambhampah)

f has a local min at $c \Leftrightarrow f'(c) = 0$, $f''(c) < 0$

$\Leftrightarrow g'(c) = 0$, $-g''(c) < 0 \Leftrightarrow g'(c)$, $g''(c) > 0$

$\Leftrightarrow g$ has a local max at c.