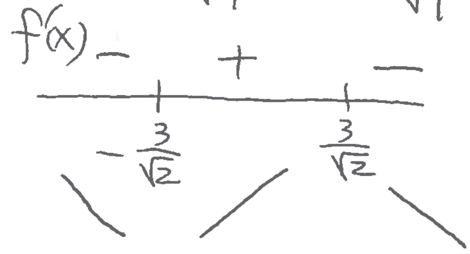
Honor Calculus, Math1450, — Midterm 1 (sample 2) — Solutions

(1) Given $f(x) = x\sqrt{9-x^2}$ on $[-3,3]$. Find abs. extreme and critical points

$$f'(x) = \sqrt{9-x^2} + x\cdot(-2x)\cdot\frac{1}{2}\frac{1}{\sqrt{9-x^2}} = \sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}} = \frac{9-2x^2}{\sqrt{9-x^2}}$$

- $f'(x) = 0 \Rightarrow 9-2x^2 = 0 \Rightarrow x = \pm\frac{3}{\sqrt{2}}$.
- $f'(x)$ DNE $\Rightarrow 9-x^2 = 0 \Rightarrow x = \pm 3$



$$f\left(\frac{3}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}\sqrt{9-\frac{9}{2}} = \frac{3}{\sqrt{2}}\cdot\frac{9}{2} = \frac{27}{4}\sqrt{2} \rightarrow \text{local max.}$$

$$f\left(-\frac{3}{\sqrt{2}}\right) = -\frac{3}{\sqrt{2}}\sqrt{9-\frac{9}{2}} = -\frac{27}{4}\sqrt{2} \rightarrow \text{local min.}$$

Endpoint. $f(3) = 0$, $f(-3) = 0$

$\Rightarrow$ Critical points: $\pm\frac{3}{2}\sqrt{2}$, ± 3; abs. max $f\left(\frac{3}{\sqrt{2}}\right)$, abs. min $f\left(-\frac{3}{\sqrt{2}}\right)$.

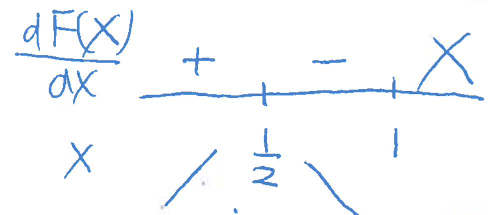
(2) Given $x \geq 0$, $y \geq 0$, $x+y=1$. Find $\max_{(x,y)}\{-x\ln(x) - y\ln(y)\}$

(a)

$\Rightarrow y = 1-x \Rightarrow$ Let $F(x) = -x\ln(x) - y\ln(y) = -x\ln(x) - (1-x)\ln(1-x)$

$$\frac{dF(x)}{dx} = -\ln(x) - 1 + \ln(1-x) + 1 = -\ln(x) + \ln(1-x) = \ln\left(\frac{1-x}{x}\right)$$

$$\frac{dF(x)}{dx} = 0 \Rightarrow \frac{1-x}{x} = 1 \Rightarrow x = \frac{1}{2}$$
((local max))



$\frac{dF(x)}{dx}$ DNE $\Rightarrow x = 1$, $\cancel{x=0}$ ($x=0$ is not in the domain of ln. functi

Since $x+y=1$, we only can have $0<x, y<1$ because of the property of ln. function.

$\Rightarrow F(\frac{1}{2})$ is the maximum value, which is

$$-\frac{1}{2}\ln(\frac{1}{2})-\frac{1}{2}\ln(\frac{1}{2})=-\ln(\frac{1}{2})=\ln(2).$$

(b). Skip!

(3) Given $\frac{y^2}{4}+x^2=1$, We have

$$\frac{\cancel{2}y}{\cancel{4}2}\cdot\frac{dy}{dt}+2x\cdot\frac{dx}{dt}=0 \quad -(*)$$

To find the points on the ellipse where $\frac{dy}{dt}=\frac{dx}{dt}$,

we have, by $(*)$,

$$\frac{dy}{dt}=-\frac{2x}{y}\cdot 2\frac{dx}{dt} \Rightarrow -\frac{4x}{y}=1 \Rightarrow -4x=y$$

put this condition back to ellipse, we have

$$\frac{(-4x)^2}{4}+x^2=1 \Rightarrow 5x^2=1 \Rightarrow x^2=\frac{1}{5}, \; x=\pm\frac{1}{\sqrt{5}}$$

$$x=\frac{1}{\sqrt{5}}, \; y=-\frac{4}{\sqrt{5}} \quad \text{or} \quad x=-\frac{1}{\sqrt{5}}, \; y=\frac{4}{\sqrt{5}}$$

(4)

(a)

(i) $\lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right)^4 = \left(\lim_{x\to 0}\frac{\sin(3x)}{x}\right)^4 = (3)^4 = 81$

(ii) $\lim_{x\to\infty}\frac{x^2+\sqrt{x}}{x^2+x^{\frac{3}{2}}+1} \underset{\text{leading coefficient}}{=} \frac{1}{1} = 1$

(iii) $\lim_{x\to 0} e^x \sin(x) = 1\cdot 0 = 0$

(iv) $\lim_{x\to\infty} e^{-\sqrt{x}}x^2 \overset{(0\cdot\infty)}{\underset{(L')}{=}} \lim_{x\to\infty}\frac{x^2}{e^{\sqrt{x}}} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}\frac{2x}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}} = \lim_{x\to\infty}\frac{4x^{\frac{3}{2}}}{e^{\sqrt{x}}}$

$\overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}\frac{6x^{\frac{1}{2}}}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}} = \lim_{x\to\infty}\frac{12x}{e^{\sqrt{x}}} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}\frac{12}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}} = \lim_{x\to\infty}\frac{24\sqrt{x}}{e^{\sqrt{x}}}$

$\overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} \lim_{x\to\infty}\frac{12\frac{1}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}} = \lim_{x\to\infty}\frac{24}{e^{\sqrt{x}}} = 0$

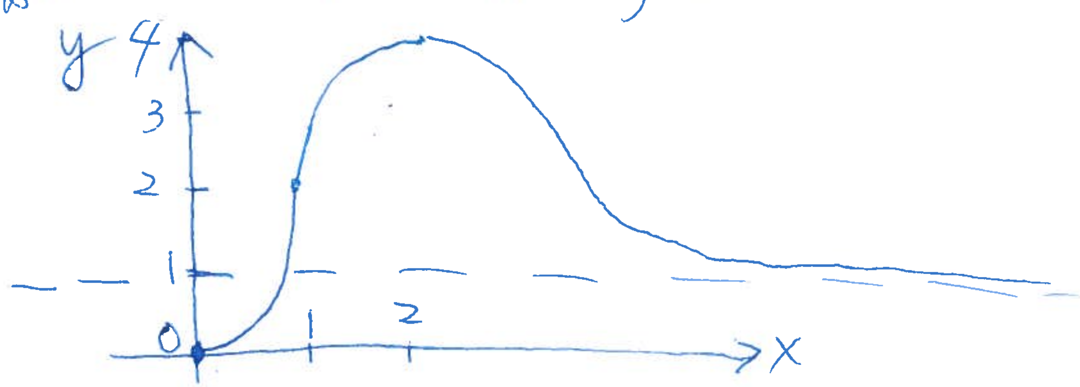
(b) ① $f(0)=0$: global min.

② f is concave up on $[0,1)$, P.O.I : $x=1$

③ $f(2)=4$: global max

④ increasing on $[0,2)$, decreasing on $(2,\infty)$

⑤ $\lim_{x\to\infty} f(x) = 1 \Rightarrow$ horizontal asym.

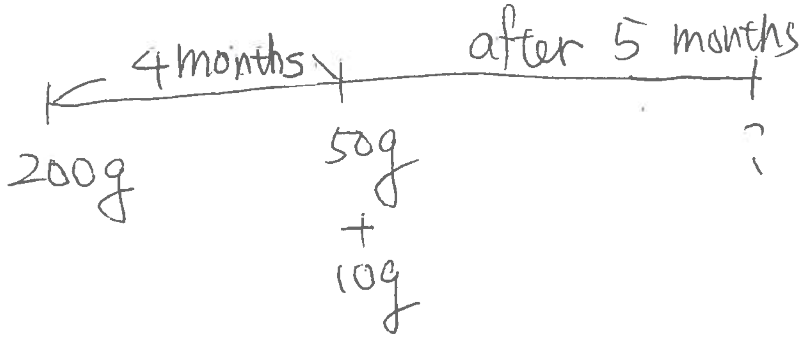


(5) The rate of change of mass P $=$ $K \cdot P(t)$ ← the mass @ time t

(a) ↑ proportional ⇒ constant K

$$\Rightarrow \frac{dP}{dt} = K \cdot P(t) \quad \text{and the solution is } P(t) = P(0)\,e^{Kt}.$$

(b)



time	0	4
New time		5
		0

$P(0) = 200$
$P(4) = 50$

$$\Rightarrow 50 = P(4) = P(0)\,e^{4K} = 200 \cdot e^{4K}$$

$$\Rightarrow 0.25 = e^{4K} \Rightarrow \ln(0.25) = 4K$$

$$\Rightarrow K = \frac{\ln(0.25)}{4} \quad \text{or} \quad K = \ln(0.25)^{\frac{1}{4}} = \ln(0.5)^{\frac{1}{2}}$$

$P(0) = 60$
$P(5) = ?$

$$\Rightarrow P(5) = P(0)\,e^{[\ln(0.5)^{\frac{1}{2}}]\cdot 5} = 60\,e^{[\ln(0.5)^{\frac{1}{2}}]\cdot 5}$$

same K

$$= 60 \cdot (0.5)^{\frac{5}{2}}$$

$$= 15\sqrt{0.5} = \frac{15}{\sqrt{2}}$$