Honor Calculus, Math 1450, — Midterm I (sample 1) solutions

(1) Given $f(x)=\frac{x+1}{x^2+1}$ on $[-1,\frac{1}{2}]$. Find local and abs. extreme.

First, to find local extreme, we have

$$f'(x)=\frac{1\cdot(x^2+1)-(x+1)(2x)}{(x^2+1)^2}=\frac{-x^2-2x+1}{(x^2+1)^2},\quad \text{since } (x^2+1)^2>0 \ \forall x\in\mathbb{R},$$

we only need to check x such that $f'(x)=0 \Leftrightarrow +x^2+2x-1=0$

$$\Rightarrow x=\frac{-2\pm\sqrt{8}}{2} \quad \left(\text{but } \frac{-2-\sqrt{8}}{2}\notin[-1,\tfrac{1}{2}]\right) \Rightarrow x=\frac{-2+\sqrt{8}}{2}=-1+\sqrt{2}$$

x	-1		$-1+\sqrt{2}$		$\frac{1}{2}$
$f'(x)$		$+$		$-$	

$$\left(\frac{\sqrt{2}}{4-2\sqrt{2}}=\frac{\sqrt{2}(4+2\sqrt{2})}{(4-2\sqrt{2})(4+2\sqrt{2})}=\frac{4\sqrt{2}+4}{8}=\frac{\sqrt{2}+1}{2}=\frac{1.41421+1}{2}=1.207105\right)$$

$$\Rightarrow f(-1+\sqrt{2})=\frac{\sqrt{2}}{4-2\sqrt{2}} \text{ is a local max.}$$

Second, check the endpoints of the given interval.

$$f(-1)=\frac{-1+1}{1+1}=0,\quad f(\tfrac{1}{2})=\frac{\frac{1}{2}+1}{\frac{1}{4}+1}=\frac{\frac{3}{2}}{\frac{5}{4}}=\frac{12}{10}=\frac{6}{5}<f(-1+\sqrt{2})$$

$\Rightarrow f(-1)$ is an abs. min and $f(-1+\sqrt{2})$ is an abs. max.

(2) Given $f(x)=2x^3-9x^2+12x+8$.

(a), (b). $f'(x)>0 \Rightarrow f$ is increasing ; $f'(x)<0 \Rightarrow f$ is decreasing

$$f'(x)=6x^2-18x+12=6(x^2-3x+2)=6(x-1)(x-2)$$

		1		2	
$f'(x)$	$+$		$-$		$+$
$(x-1)$	$-$		$+$		$+$
$(x-2)$	$-$		$-$		$+$

$\Rightarrow f'(x)>0$, increasing intervals $x\in(-\infty,1)\cup(2,\infty)$

and $f'(x)<0$, decreasing interval $x\in(1,2)$

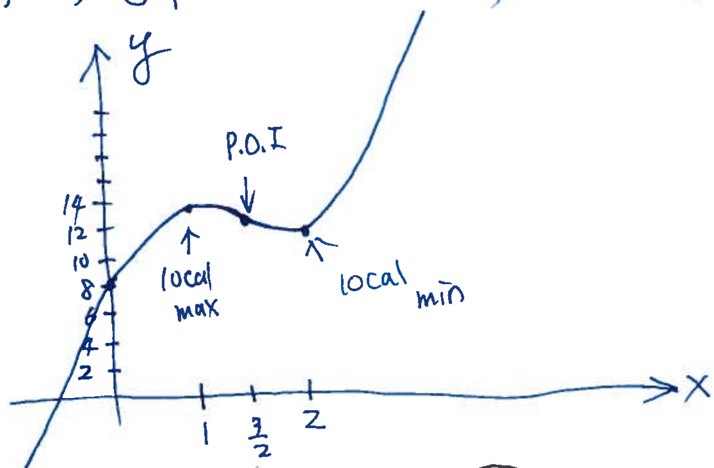
(2) (c)(d) $f''(x)>0 \Rightarrow f$ is concave up ; $f''(x)<0 \Rightarrow f$ is concave down.

$f''(x) = 12x - 18 \Rightarrow f''(x)$

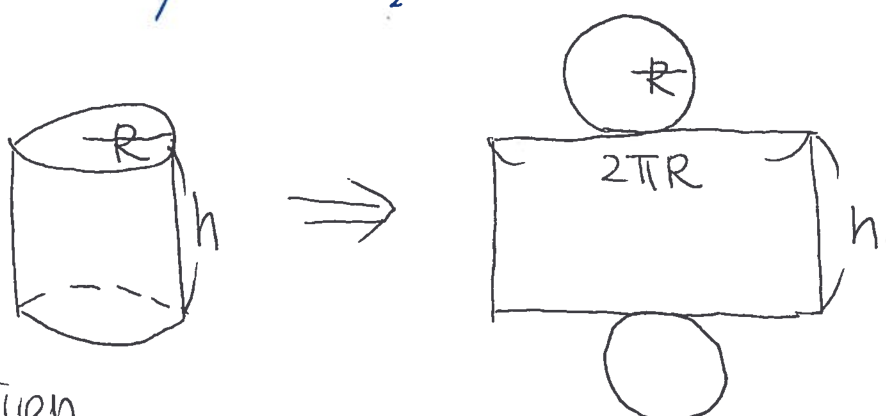
$\Rightarrow f''(x)>0$ concave up interval $\left(\frac{3}{2}, \infty\right)$

$f''(x)<0$ concave down interval $\left(-\infty, \frac{3}{2}\right)$

Graph: $f(0)=8$, $f(1)=13$, $f(2)=12$, $f\left(\frac{3}{2}\right)=\frac{25}{2}$



(3)



Given

$R^2\pi h = 1600\ (cm)^3 \Rightarrow$ Find the minimum of surface:

$S = 2\pi R^2 + 2\pi R h.$

$\rightarrow h = \frac{1600}{R^2\pi} \rightarrow S = 2\pi R^2 + 2\cancel{\pi}\cancel{R}\frac{1600}{R^{\cancel{2}}\cancel{\pi}}$

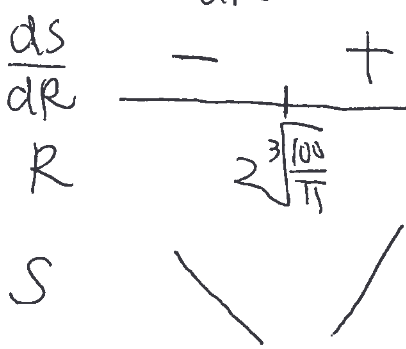
$= 2\pi R^2 + \frac{3200}{R}$

(3) conti.

$$\frac{dS}{dR} = 4\pi R + \frac{-3200}{R^2} = \frac{4\pi R^3 - 3200}{R^2} \Rightarrow$$

if $\frac{dS}{dR} = 0$, then $R^3 = \frac{\cancel{3200}^{\,800}}{\cancel{4}\pi} \Rightarrow R = 2\sqrt[3]{\frac{100}{\pi}}$

(if $\frac{dS}{dR}$ DNE $\Rightarrow R=0$. but, obviously, we don't need this condition)



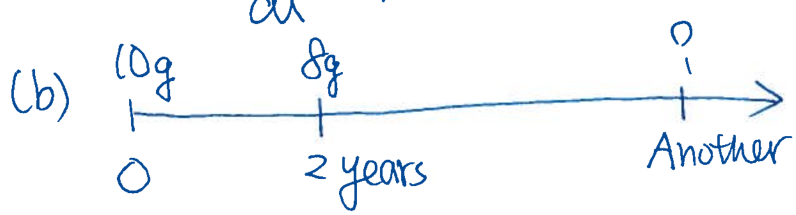
$\Rightarrow$ local min. as $R = 2\sqrt[3]{\frac{100}{\pi}}$.

dimensions: $R = 2\sqrt[3]{\frac{100}{\pi}}$, $h = \frac{40}{\sqrt[3]{10\pi}}$

(4)
(a) Rate of decay of uranium proportional to the mass of uranium.

Let m be the mass of uranium, then.

$$\frac{dm}{dt} = k\,m(t). \Longrightarrow\Longrightarrow m(t) = e^{kt}\cdot m(0)$$

(b)


$\Rightarrow m(0) = 10$ and $m(2) = m(0)\cdot e^{k\cdot 2} = 8$.

$\Rightarrow 8 = 10\cdot e^{2k} \Rightarrow k = \frac{1}{2}\ln(0.8)$

$$m(5) = m(0)\cdot e^{5\cdot\frac{1}{2}\ln(0.8)} = 10\cdot e^{\ln(0.8)^{\frac{5}{2}}} = 10\cdot(0.8)^{\frac{5}{2}}\ \text{(grams)}.$$

(5) f is differentiable on $(-2,6)$, $f(-1)=1$, and $-3\le f'(x)\le 3$ for all $x\in(-1,2)$. To show $-5\le f(1)\le 7$.

By mean value theorem, we have

$$\exists\, c\in(-1,1) \text{ such that } f'(c)=\frac{f(1)-f(-1)}{2}=\frac{f(1)-1}{2}.$$

$$\Rightarrow \quad f(1)=2f'(c)+1.$$

Since $-3\le f'(c)\le 3$, $c\in(-1,2)$, then

$$2(-3)+1 \le f(1)\le 2(3)+1 \Rightarrow -5\le f(1)\le 7.$$

(6)

(a)

① rest at a height 2000m above the ground $\Rightarrow x_1(0)=2000$

free fall $\Rightarrow$ initial velocity $=0 \Rightarrow \dot{x}_1(0)=0$

$$m\ddot{x}_1=-mg \Rightarrow \ddot{x}_1=-g \Rightarrow \dot{x}_1=-gt+\dot{x}_1(0)$$
$$=-gt$$

$$\Rightarrow x_1=-\frac{1}{2}gt^2+x_1(0)=-\frac{1}{2}gt^2+2000.$$

② Hit the ground $\Rightarrow x_1=0$ find t?

$$0=-\frac{1}{2}\cdot 10\cdot t^2+2000 \Rightarrow t^2=400 \Rightarrow t=20\,(s).$$

(6) (b) ① fired upward from ground $\Rightarrow x_2(0)=0$

with a velocity $V_0 \Rightarrow \dot{x}_2(0)=V_0$.

$$\ddot{x}_2=-g \Rightarrow \dot{x}_2=-gt+\dot{x}_2(0)=-gt+V_0.$$

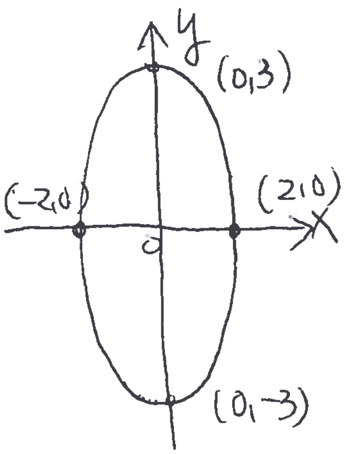
$$\Rightarrow x_2=-\frac{1}{2}gt^2+V_0t+x_2(0)=\underline{-\frac{1}{2}gt^2+V_0t}$$

② projectile hits the object in the air $\Rightarrow x_1(t)=x_2(t)$.

$$\Rightarrow -\frac{1}{2}gt^2+2000=-\frac{1}{2}gt^2+V_0t \Rightarrow V_0t=2000.$$

(7)

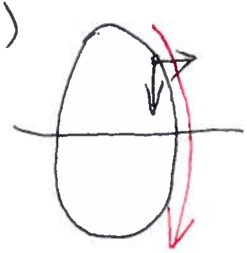
(a)



$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

(b) at point $(1, \frac{\sqrt{27}}{2})$ x-coordinate of particle increases at a rate of 1 cm/s $\Rightarrow \left.\frac{dx}{dt}\right|_{(x,y)=(1,\frac{\sqrt{27}}{2})}=1.$

(c)



$$\frac{x^2}{4}+\frac{y^2}{9}=1 \xRightarrow{\frac{d}{dt}} \frac{2x}{4}\frac{dx}{dt}+\frac{2y}{9}\frac{dy}{dt}=0$$

at $(1,\frac{\sqrt{27}}{2})$ $\Rightarrow$ $\frac{1}{2}\cdot 1+\frac{\sqrt{27}}{9}\cdot\frac{dy}{dt}=0 \Rightarrow \frac{dy}{dt}=-\frac{1}{2}\cdot\frac{9}{\sqrt{27}}<0$

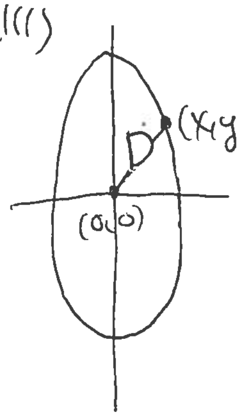
$\Rightarrow$ <u>clockwise.</u>

(7) conti.

(b)

(ii) $\frac{dy}{dt} = -\frac{1}{2} \cdot \frac{9}{\sqrt{27}} = -\frac{1}{2} \cdot \frac{3}{\sqrt{3}} = -\frac{\sqrt{3}}{2}$ (cm/s).

(iii)



$\left.\frac{dD}{dt}\right|_{(1,\frac{\sqrt{27}}{2})} = ?$

$D = \sqrt{x^2+y^2} \Rightarrow \frac{dD}{dt} = \left(2x \cdot \frac{dx}{dt} + 2y \frac{dy}{dt}\right) \cdot \frac{1}{\sqrt{x^2+y^2}}$

$\Rightarrow \left.\frac{dD}{dt}\right|_{(1,\frac{\sqrt{27}}{2})} = \left(2 \cdot 1 \cdot 1 + 2 \cdot \frac{\sqrt{27}}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right)\right) \cdot \frac{1}{\sqrt{1+\left(\frac{\sqrt{27}}{2}\right)^2}}$

$= \left(2 - \frac{9}{2}\right) \cdot \frac{2}{\sqrt{31}} = -\frac{5}{\sqrt{31}}$ cm/s