Honors Calculus, another sample final — solution.

(1) See the solution of sample I and sample 2.

(2) $\sum_{n=3}^{\infty} \frac{n^{\frac{1}{3}}+7}{n^{\frac{3}{2}}+2n+3}$

(a) Let $a_n = \frac{n^{\frac{1}{3}}+7}{n^{\frac{3}{2}}+2n+3}$, $b_n = \frac{n^{\frac{1}{3}}}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{7}{6}}}$. we have $\sum b_n$ converges by p-series and

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{n^{\frac{1}{3}}+7}{n^{\frac{3}{2}}+2n+3} \cdot \frac{n^{\frac{3}{2}}}{n^{\frac{1}{3}}}\right| = \left|\frac{n^{\frac{11}{6}}+7n^{\frac{3}{2}}}{n^{\frac{11}{6}}+2n^{\frac{4}{3}}+3n^{\frac{1}{3}}}\right| \to 1 \text{ as } n\to\infty$$

where $1<\infty$ and $1>0$. Then, by L.C.T., $\sum_{n=3}^{\infty} \frac{n^{\frac{1}{3}}+7}{n^{\frac{3}{2}}+2n+3}$ converges.

(b) $\sum_{n=2}^{\infty} \frac{2}{n(\ln(n))}$

(let $u=\ln(x)$, $du = \frac{dx}{x}$)

Let $f(x) = \frac{2}{x(\ln(x))}$. Then $\int_2^{\infty} \frac{2\,dx}{x(\ln(x))} = 2\int_2^{\infty} \frac{1}{\ln(x)}\frac{dx}{x} = 2\ln(\ln(x))\Big|_2^{\infty} \to \infty$

By Integral test, $\sum_{n=2}^{\infty} \frac{2}{n(\ln(n))}$ diverges.

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Let $a_n = \frac{2^n}{n!}$. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right| = \left|\frac{2}{n+1}\right| \to 0 < 1$ as $n\to\infty$.

By Ratio Test, $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

(d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

Let $b_n = \frac{1}{\sqrt{n}}$. Since (1) $b_n > b_{n+1} \geq 0$ $\forall n$ (2) $b_n \to 0$ as $n\to\infty$.

Then, by A.S.T., $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

(3)

(a) Given $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 2^n}$, let $a_n = \frac{(x-1)^n}{n^2 2^n}$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)^{n+1}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 2^n}{(x-1)^n}\right| = \left|\frac{x-1}{2} \cdot \left(\frac{n}{n+1}\right)^2\right| \to \frac{x-1}{2} \quad \text{as } n\to\infty$$

By Ratio Test, $\left|\frac{x-1}{2}\right| < 1 \Rightarrow |x-1| < \underset{\text{radius.}}{\underset{\uparrow}{2}} \Rightarrow -2 < x-1 < 2$

$\Rightarrow -1 < x < 3$

check $x=-1$, we have $\sum \frac{(-2)^n}{n^2 \cdot 2^n} = \sum \frac{(-1)^n}{n^2}$ converges by A.S.T.

check $x=3$ we have $\sum \frac{2^n}{2^n \cdot n^2} = \sum \frac{1}{n^2}$ converges by p-series Test.

Thus, the radius of convergence is 2 and interval of convergence is $-1 \le x \le 3$.

(b) Given $f(x) = \sum_{n=1}^{\infty} n x^{n-1}$.

(i) let $a_n = n x^{n-1}$. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)x^n}{n\ x^{n-1}}\right| = |x| \cdot \left|\frac{n+1}{n}\right| \to |x|$ as $n\to\infty$,

By Ratio Test, as $|x|<1$ (or $-1<x<1$). $\sum_{n=1}^{\infty} n x^{n-1}$ converges,

check $x=1$, we have $\sum_{n=1}^{\infty} n$ diverges by Divergence Test.

check $x=-1$. we have $\sum_{n=1}^{\infty} (-1)^{n-1} n$ diverges by A.S.T.

$\Rightarrow -1 < x < 1$. such that $\sum_{n=1}^{\infty} n x^{n-1}$ converges.

(ii) $f'(x) = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$

let $b_n = n(n-1)x^{n-2}$, $\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(n+1)n \cdot x^{n-1}}{n(n-1)x^{n-2}}\right| = |x| < 1$

check $x=1$, $\sum_{n=2}^{\infty} n(n-1)$ diverges by Divergence Test,

check $x=-1$, $\sum_{n=2}^{\infty} (-1)^{n-2} n(n-1)$ diverges by A.S.T.

as $|x|<1$, $\sum_{n=2}^{\infty} n(n-1)x^{n-2}$ converges.

Thus $f(x)$ is differentiable as $|x|<1$.

(4) since

(a) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Then $\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$.

(b) T_4 of $\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}$ and

$|R_4(x)| \leq \frac{M \cdot |x|^5}{5!}$

(formula, $|R_n(x)| \leq \frac{M}{(n+1)!} \cdot |x-a|^{n+1}$ for $|f^{(n+1)}(x)| \leq M$ as $|x-a| \leq d$).

Since $|\cos^{(n+1)}(x)| \leq 1$, let $M=1$. as $|x| < \frac{1}{2}$

we have $|R_4(x)| \leq \frac{1 \cdot |x|^5}{5!} \leq \frac{1}{2^5 \cdot 5!}$

(c) By the remainder formula. for fixed $x \in \mathbb{R}$, $M=1$..

$\frac{M \cdot |x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$.

So, as $n \to \infty$, $\cos(2x) = \lim_{n\to\infty} T_n = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.

(5).

let $f(x) = \frac{1}{1-2x^3}$,

we have $\frac{1}{1-2x^3} = \frac{1}{1-(2x^3)} = \sum_{n=0}^{\infty} (2x^3)^n$

(i) let $a_n = (2x^3)^n$, by Root Test, we have

$\sqrt[n]{|a_n|} = |2x^3| < 1 \Rightarrow |x^3| < \frac{1}{2} \Rightarrow \frac{-1}{\sqrt[3]{2}} < x < \frac{1}{\sqrt[3]{2}}$

($-\frac{1}{2} < x^3 < \frac{1}{2}$, as $x^3 = \frac{1}{2} \Rightarrow x = \frac{1}{\sqrt[3]{2}}$, as $x^3 = \frac{-1}{2}$, $x = -\frac{1}{\sqrt[3]{2}}$)

check $x = \frac{1}{\sqrt[3]{2}}$, $\sum_{n=0}^{\infty} (2)^n \cdot \left(\frac{1}{\sqrt[3]{2}}\right)^{3n} = \sum_{n=0}^{\infty} 1$ diverges, by Divergence Test,
($1 \not\to 0$) as $n \to \infty$

check $x = \frac{-1}{\sqrt[3]{2}}$, $\sum_{n=0}^{\infty} (2)^n \left(\frac{-1}{\sqrt[3]{2}}\right)^{3n} = \sum_{n=0}^{\infty} (-1)^n$ diverges by Divergence Test
($(-1)^n \not\to 0$ as $n \to \infty$)

$\Rightarrow$ As $\frac{-1}{\sqrt[3]{2}} < x < \frac{1}{\sqrt[3]{2}}$, $\sum_{n=0}^{\infty} (2x^3)^n$ converges.

(ii)
$$f(x) = \sum_{n=0}^{\infty} (2x^3)^n = 1 + 2x^3 + (2x^3)^2 + \cdots$$
$$= 1 + 2x^3 + 4x^6 + \cdots$$

Compare with the Taylor expansion, we have

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(6)}(0)}{6!}x^6 + \cdots \text{ and}$$

$$4x^6 = \frac{f^{(6)}(0)}{6!}x^6 \Rightarrow f^{(6)}(0) = 4 \cdot 6!$$