Honors Calculus, another sample final.

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ATTEMPT ALL QUESTIONS. SHOW ALL WORKING. POINTS WILL NOT BE AWARDED IF WORKING IS NOT SHOWN. NO PROGRAMMABLE CALCULATORS ARE TO BE USED.

Please write your answers clearly and in a logical and well-organized way. Points will be deducted for sloppy work.

It is helpful in the problems below to recall that:

- $\lim_{n\to\infty} \frac{\log n}{n^\alpha} = 0$ for any $\alpha > 0$ and
- $\lim_{n\to\infty} \gamma^n n^\alpha = 0$ for any $\alpha > 0$ and any $|\gamma| < 1$
- $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$ for any α

i.e. a factorial dominates an exponential, an exponential dominates a polynomial and a polynomial dominates a logarithm.

GOOD LUCK!

(1) (10 points) In a page or less describe what it means for a series

$$\sum_{n=0}^{\infty} a_n$$

to be: convergent; divergent; absolutely convergent and conditionally convergent. Make sure to mention the role of partial sums and illustrate your discussion with examples.

(2) (20 points) Determine whether or not the following series converge. State precisely your reasons.

(a)

$$\sum_{n=3}^{\infty} \frac{n^{1/3}+7}{n^{3/2}+2n+3}$$

(b)

$$\sum_{n=2}^{\infty} \frac{2}{n\ln(n)}$$

(c)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

(d)

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

(3) (10 points) (a) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 2^n}$$

and find all points at which this power series converges.

(b) Suppose we define a function $f(x)$ by

$$f(x) = \sum_{n=1}^{\infty} n x^{n-1}$$

at all points where the power series converges.

(i) Find all points where the power series converges.

(ii) Is $f(x)$ differentiable at the points where the power series converges? If it is differentiable, give reasons why it is differentiable and find its derivative as a power series. If it is not differentiable, briefly say why not.

(4) (10 points) (a) Find the Taylor series for $\cos(2x)$ about $a = 0$.

(b) Find the Taylor polynomial approximation $T_4(x)$ of degree 4 to $\cos(2x)$ and estimate the accuracy of the approximation for $|x| < .5$.

(c) By estimating the remainder term $R_n(x) = \cos(2x) - T_n(x)$ show that $\cos(2x)$ equals its Taylor series for all values of x.

(5) (10 points) Find the power series expansion of the function

$$\frac{1}{1-2x^3}$$

about $a = 0$.

(i) For what values of x does the power series converge?

(ii) Define

$$f(x) = \frac{1}{1-2x^3}$$

for $x \neq 2^{-1/3}$ and using (i) or otherwise find $f^{(6)}(0)$.