Honors Calculus, Sample Final — Solution

(1) Given $\sum_{n=0}^{\infty} a_n (x-a)^n$, R is its radius of convergence.

If $x \in (a-R, a+R)$, $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges.

⋮

(2)

(i) This statement is true since $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = 0$ implies

$|a_n| < |b_n|$ as n is large enough and $\sum_{n=0}^{\infty} |b_n|$ converges

So, by B.C.T, $\sum_{n=0}^{\infty} |a_n|$ converges. So do $\sum_{n=0}^{\infty} a_n$

(since $a_n \le |a_n|$)

Example: $a_n = \frac{\ln(n)}{n^3}$ and $b_n = \frac{1}{n^2}$, $|a_n| < |b_n|$ and

we have $\left|\frac{a_n}{b_n}\right| = \left|\frac{\ln(n)}{n^3} n^2\right| = \left|\frac{\ln(n)}{n}\right| \to 0$ as $n \to \infty$.

(ii) This statement is false.

Counterexample: Let $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$, we have

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{\frac{1}{n^2}}{\frac{1}{n}}\right| = \left|\frac{1}{n}\right| \to 0 \text{ as } n \to \infty \text{ and } \sum \left|\frac{1}{n^2}\right| \text{ converges.}$$

but $\sum \frac{1}{n}$ diverges.

2. (iii) This statement is false.

Counterexample: let $a_n = \frac{1}{n}$

$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} \frac{1}{n^2} = 0$ but $\Sigma a_n = \Sigma \frac{1}{n}$ diverges.

3. (a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Let $a_n = \frac{n^2}{2^n}$, then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right| = \left|\frac{1}{2} \cdot \frac{(n+1)^2}{n^2}\right| \to \frac{1}{2} < 1$ as $n\to\infty$

Then, by Ratio Test, $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(b) $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$

Since $\lim_{n\to\infty} \frac{\ln(n)}{n^{\frac{1}{2}}} = 0 \Rightarrow \ln(n) < n^{\frac{1}{2}}$ as n is large enough

$\Rightarrow \frac{1}{n^{\frac{1}{2}}} < \frac{1}{\ln(n)} \Rightarrow \frac{1}{n} < \frac{1}{(\ln(n))^2}$.

Then, since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we have $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$ diverges by B.C.T.

(c) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

Let $a_n = \frac{2^n n!}{n^n}$, we have $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!}\right|$

$= \left|\frac{2\cdot(n+1)}{(n+1)} \frac{n^n}{(n+1)^n}\right| = \left|\frac{2n^n}{(n+1)^n}\right|$.

3. (c) (conti.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|2 \cdot \underbrace{\left(\frac{n}{n+1}\right)^n}_{\to \frac{1}{e} \text{ as } n\to\infty}\right| \to \frac{2}{e} < 1 \quad \text{as } n\to\infty.$$

So, by ratio Test, $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ converges.

(d). $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Let $a_n = \frac{1}{n(1+n)}$, $b_n = \frac{1}{n^2}$, $\left|\frac{a_n}{b_n}\right| = \left|\frac{n^2}{n(1+n)}\right| \to 1$ as $n\to\infty$

Since 1 is finite and positive, then, by L.C.T. (Limit Comparison Test), $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

4. (i) let $f(x) = e^x \cos(x)$. Find T_4 of $f(x)$.

we have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ and

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

Then $f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$

$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x - \frac{x^3}{2!} + \frac{x^5}{3!} + \frac{x^2}{2!} - \frac{x^4}{2!2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$

$= 1 + x + \left(-\frac{1}{2!} + \frac{1}{3!}\right)x^3 + \left(\frac{2}{4!} - \frac{1}{2!2!}\right)x^4 + \cdots$

$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$

$\Rightarrow T_4 = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$.

4. (ii) By formula, we have $|R_n(x)| \leq \dfrac{M|x-a|^{n+1}}{(n+1)!}$

where $|f^{(n+1)}(x)| \leq M$.

Let $f(x)=e^x$, and $f^{(n+1)}(x)=e^x$, $a=0$

(a) as $n=3$, we have

$$|R_3(x)| \underset{\substack{\uparrow \\ c\in(0,1)}}{=} \left|\frac{e^c x^{3+1}}{(3+1)!}\right| \overset{\substack{|x|<1 \\ \downarrow}}{\leq} \frac{e^1 \cdot x^{3+1}}{(3+1)!} \leq \frac{e^1 \cdot 1^4}{4!} = \frac{e}{24}.$$

(b)

Since $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ and, let $a_n = \dfrac{x^n}{n!}$,

we have $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \left|\dfrac{x}{n+1}\right| \to 0^{<1}$ as $n \to \infty$

So, by Ratio Test, the radius of convergence is ∞.

and $x \in (-\infty, \infty)$ such that $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ converges.

and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ for all $x \in \mathbb{R}$ (or $x \in (-\infty, \infty)$).

5.

(i) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$.

(ii) let $f(x) = \frac{1}{(1-x)^2}$. we have $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

Then $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$.

For $g(x) = \frac{1}{(1-x)^3}$ we have $\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{+2}{(1-x)^3}$

Then $\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{1}{2}\cdot\sum_{n=2}^{\infty} n(n-1)\, x^{n-2}$.