Honors Calculus, Sample Final.

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ATTEMPT ALL QUESTIONS. SHOW ALL WORKING. POINTS WILL NOT BE AWARDED IF WORKING IS NOT SHOWN. NO PROGRAMMABLE CALCULATORS ARE TO BE USED. TIME ALLOWED: 80 MINUTES

Please write your answers clearly and in a logical and well-organized way. Points will be deducted for sloppy work.

GOOD LUCK!

(1) [15 points] Suppose

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

is a power series. In a page or less describe the radius of convergence R for a power series and the behavior of the function

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

on the interval $(a-R, a+R)$ in regard to integrability and differentiability.

Given a function $f(x)$ how do we determine a power series expansion for f about a point $x = a$?

Illustrate your discussion with examples.

(2) [10] Explain the truth or falsity of the following two statements, giving reasons and examples:

(i) If $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = 0$ and $\sum_{n=0}^{\infty} |b_n|$ converges then $\sum_{n=0}^{\infty} a_n$ converges.

(ii) If $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = 0$ and $\sum_{n=0}^{\infty} |a_n|$ converges then $\sum_{n=0}^{\infty} b_n$ converges.

(iii) If $\lim_{n\to\infty} a_n^2 = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

(3) [20 points] Determine whether the following series converge. State precisely your reasons.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$$

Hint: $\lim_{n\to\infty} \frac{\ln(n)}{n^p} = 0$ *for any* $p > 0$.

(c)

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$$

(d)

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots$$

(4) [15 points] (i) Find the 4th order Taylor polynomial about $a = 0$ of the function

$$e^x \cos x$$

(ii) Let $T_n(x)$ be the nth order Taylor polynomial and $R_n(x)$ be the nth order Taylor remainder for the function e^x about $a = 0$.

By estimating the remainder term $R_n(x) = e^x - T_n(x)$:

(a) estimate the error when approximating e^x by $T_3(x)$ for $|x| < .1$.

(b) show that e^x equals its Taylor series for all values of x.

(5) [10 points] (i) Find the power series expansion of the function

$$\frac{1}{1-x}$$

about $a = 0$.

(ii) Using (i) or otherwise find the Taylor series expansion of

$$\frac{1}{(1-x)^2}$$

and

$$\frac{1}{(1-x)^3}$$

about $a = 0$, stating carefully any theorems you may use about integrating or differentiating power series within their radius of convergence.