Honor Calculus. Sample Final 1 – Solution.

(1) (a) Let $S_k = \sum_{n=1}^{k} a_n$

We say $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{k\to\infty} S_k$ exists.

If NOT, we say $\sum_{n=1}^{\infty} b_n$ diverges.

(b) For a conditionally convergent series $\sum_{n=1}^{\infty} a_n$,

$\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is NOT.

For a absolutely convergent series $\sum_{n=1}^{\infty} b_n$,

both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |b_n|$ are convergent.

For example, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent,

since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by A.S.T. but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ is divergent by P-series Test.

(2) (a) $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}+1}{2^n}$

Let $a_n = \frac{n^2+\sqrt{n}+1}{2^n}$ and $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2+\sqrt{n+1}+1}{2^{n+1}} \cdot \frac{2^n}{n^2+\sqrt{n}+1}\right|$

$$= \left|\frac{1}{2}\cdot\frac{n^2+2n+1+\sqrt{n+1}+1}{n^2+\sqrt{n}+1}\right| \to \frac{1}{2} < 1$$

as $n\to\infty$.

Then, by Ratio Test, $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}+1}{2^n}$ converges.

(2)

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\,\ln(n)}$

Since $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = 0$,

Then $\ln(n) < \sqrt{n}$ for a large $n$.

Thus $\displaystyle \frac{1}{\sqrt{n}} < \frac{1}{\ln(n)} \Rightarrow \frac{1}{n} = \frac{1}{\sqrt{n}\cdot\sqrt{n}} < \frac{1}{\sqrt{n}\ln(n)}$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\cdot\ln(n)}$ is divergent.
by Basic Comparison Test
(B.C.T)

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\sqrt{2}}\ln(n)}$

Since $\displaystyle \frac{1}{n^{\sqrt{2}}\ln(n)} < \frac{1}{n^{\sqrt{2}}}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\sqrt{2}}}$ converges by P-series Test
($\sqrt{2} > 1$)

Then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\sqrt{2}}\ln(n)}$ converges by B.C.T.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Let $b_n = \frac{1}{\sqrt{n}}$

Since (1) $b_n > b_{n+1} > 0$ $\forall n \in \mathbb{N}$ (for all natural number $n$)

(2) $b_n \to 0$ as $n \to \infty$.

Then, by A.S.T. (Alternating Series Test)

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Let $a_n = \frac{n^n}{n!}$. We have $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right| = \left|\frac{n+1}{n+1} \cdot \frac{(n+1)^n}{n^n}\right|$

(2) (e) (conti.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\left(\frac{n+1}{n}\right)^n\right| \to \underset{\sim}{e} \text{ as } n \to \infty$$
$$> 1$$

Then, by Ratio Test, $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

(3)

(i) $$\frac{1}{1+3x^3} = \frac{1}{1-(-3x^3)} \overset{\text{geometric series}}{=} \sum_{n=0}^{\infty} (-3x^3)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 3^n x^{3n}.$$

let $a_n = (-1)^n \cdot 3^n \cdot x^{3n}$, $\sqrt[n]{|a_n|} = \sqrt[n]{|(-1)^n 3^n x^{3n}|} = 3 \cdot x^3$

By Root test, we have $|3x^3| < 1 \Rightarrow |x^3| < \frac{1}{3}$.

$\Rightarrow \frac{-1}{\sqrt[3]{3}} < x < \frac{1}{\sqrt[3]{3}}$ $\left(\begin{array}{l} \text{as } |x^3| = \frac{1}{3} \Rightarrow x = \frac{1}{\sqrt[3]{3}} \text{ or } -\frac{1}{\sqrt[3]{3}} \\ \text{Then } |x^3| < \frac{1}{3} \text{ implies } \frac{-1}{\sqrt[3]{3}} < x < \frac{1}{\sqrt[3]{3}} \end{array}\right)$

which implies the radius of convergence is $\frac{1}{\sqrt[3]{3}}$.

Check $x = \frac{1}{\sqrt[3]{3}}$, we have $\sum_{n=0}^{\infty} (-1)^n \cdot 3^n \cdot \left(\frac{1}{\sqrt[3]{3}}\right)^{3n} = \sum_{n=0}^{\infty} (-1)^n$

which is divergent.

check $x = \frac{-1}{\sqrt[3]{3}}$, we have $\sum_{n=0}^{\infty} (-1)^n 3^n \left(\frac{-1}{\sqrt[3]{3}}\right)^{3n} = \sum_{n=0}^{\infty} 1$

which is divergent.

Thus, $\sum_{n=0}^{\infty} (-3x^3)^n$ converges as $-\frac{1}{\sqrt[3]{3}} < x < \frac{1}{\sqrt[3]{3}}$.

(3)

(ii)

By Taylor's Series of $f(x)$, we have

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(6)}(0)}{6!}x^6 + \cdots$$

Since $f(x) = \dfrac{1}{1+3x^3} = \sum_{n=0}^{\infty} (-3x^3)^n = 1 - 3x^3 + (-3x^3)^2 + \cdots$

We have $\dfrac{f^{(6)}(0)}{6!}x^6 = (-3x^3)^2 = 9x^6$

$$\Rightarrow f^{(6)}(0) = 6! \cdot 9$$

(4) Given $f(x) = \dfrac{1}{\sqrt{x}}$, $a=9$. Find $T_3$.

$f(x) = x^{-\frac{1}{2}} \Rightarrow f(9) = \dfrac{1}{3}$

$f'(x) = -\dfrac{1}{2}x^{-\frac{3}{2}} \Rightarrow f'(9) = -\dfrac{1}{2}\cdot\dfrac{1}{27} = -\dfrac{1}{54}$

$f''(x) = \dfrac{3}{4}x^{-\frac{5}{2}} \Rightarrow f''(9) = \dfrac{3}{4}\cdot\dfrac{1}{3^5} = \dfrac{1}{4\cdot 3^4}$

$f'''(x) = -\dfrac{15}{8}x^{-\frac{7}{2}} \Rightarrow f'''(9) = -\dfrac{15}{8}\cdot\dfrac{1}{3^7} = -\dfrac{5}{8\cdot 3^6}$

Then $T_3 = \dfrac{1}{3} - \dfrac{1}{54\cdot 1!}(x-9) + \dfrac{(x-9)^2}{4\cdot 3^4\cdot 2!} - \dfrac{5(x-9)^3}{8\cdot 3^6\cdot 3!}$

(5) (a) If $f(x)=\sum_{n=1}^{\infty} a_n (x-a)^n$, we have

$$f'(x)=\sum_{n=1}^{\infty} n\, a_n (x-a)^{n-1} \quad \text{and}$$

$$\int f(x)\,dx = C+\sum_{n=1}^{\infty} \frac{a_n (x-a)^{n+1}}{n+1}, \quad c \text{ is a constant.}$$

(b) Since $\int \frac{dx}{1+x^2} = \tan^{-1}(x)$,

and $\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n$, Then.

$$\tan^{-1}(x)=\int \sum_{n=0}^{\infty}(-x^2)^n dx = C+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

Since $\tan^{-1}(0)=0 \Rightarrow C=0 \Rightarrow \tan^{-1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

(6) Let $f(x)=\cos(x)$. Then the n-th Taylor Polynomial for $f(x)$

is $T_n=\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{(-1)^k x^{2k}}{(2k)!}$, and $\cos(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$

Then $R_n(x)=\cos(x)-T_n=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}-\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{(-1)^k x^{2k}}{(2k)!}$

(Here $\lfloor \frac{n}{2} \rfloor$ is an integer which is a greatest integer less than or equal to $\frac{n}{2}$) $=\sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$

(6)

① Let $f(x) = \cos(x)$. $T_n$ = nth Taylor poly. of $f(x)$, $R_n(x) = f(x) - T_n$.

By formula, $|R_n(x)| \leq \frac{M\,|x-a|^{n+1}}{(n+1)!}$ where $|f^{(n+1)}(x)| \leq M$.

Since $|\cos^{(n+1)}(x)| \leq 1$ we have, as $a=0$.

$$|R_n(x)| \underset{\substack{M=1\\ a=0}}{\underset{\uparrow}{<}} 1 \cdot \frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$$

② since $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, by ratio Test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} \cdot x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n x^{2n}}\right|$$

$$= \left|\frac{(-1)\cdot x^{2n+2}}{x^{2n}} \cdot \frac{(2n)!}{(2n+2)!}\right| = \left|\frac{x^2}{(2n+2)(2n+1)}\right| \to 0^{<1} \text{ as } n \to \infty$$

which is always less than 1, so

as $x \in (-\infty, \infty)$ $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ always converges, and

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ as $x \in (-\infty, \infty)$