Math 1432

Final Exam Review

1. Give the equation of the tangent line to the given graph at the point where x = 0
   a. $f(x) = \ln(6x+1) + e^{2x}$ $\Rightarrow$ $y-1=8x$
   b. $f(x) = \ln(2x+1) - 3e^{-4x}$ $\Rightarrow$ $y+3=14x$
   c. $f(x) = \sqrt{9-x^2}$ $\Rightarrow$ $y=3$.
2. Find the inverse of the following:
   a. $f(x) = \dfrac{2}{3-x}$ $\Rightarrow$ $y=3-\frac{2}{x}$
   b. $f(x) = \dfrac{x+1}{x+2}$ $\Rightarrow$ $y=\frac{1}{1-x}-2$
3. Find the derivative of the inverse for the following:
   a. $f(x) = x^3+1,\quad f(2)=9,\quad (f^{-1})'(9) =$ $\frac{1}{12}$
   b. $f(-3)=1,\ f(1)=2,\ f'(-3)=3,\ f'(1)=-2,\quad (f^{-1})'(1) =$ $\frac{1}{3}$
   c. $f(x)$ passes through the points (3 , -2) and (-2 , 1). The slope of the tangent line to the graph of $f(x)$ at **x** = 3 is -1/4. Evaluate the derivative of the inverse of f at -2. -4
4. Find the equation of the tangent and the normal lines to the parametric curves at the given points:

	tangent	normal
a. $x(t) = -2\cos 2t,\ y(t) = 4+2t,\ (-2,4)$	$x=-2$	$y=4$
b. $x(t) = 3\cos(3t)+2t,\ y(t) = 1+5t,\ (3,1)$	$y-1=\frac{5}{2}(x-3)$	$y-1=-\frac{2}{5}(x-3)$

5. Give an equation relating *x* and *y* for the curve given parametrically by
   a. $x(t) = -1+3\cos t \quad y(t) = 1+2\sin t$ $\quad 1=\left(\frac{x+1}{3}\right)^2+\left(\frac{y-1}{2}\right)^2$
   b. $x(t) = -1+3\cosh t \quad y(t) = 1+2\sinh t$ $\quad 1=\left(\frac{x+1}{3}\right)^2-\left(\frac{y-1}{2}\right)^2$
   c. $x(t) = -1+4e^{t} \quad y(t) = 2+3e^{-t}$ $\quad \frac{x+1}{4}=\frac{3}{y-2}$
6. Differentiate the function:
   a. $f(x) = 3^{x^2}$ $\Rightarrow f'(x)=[2x\ln 3]3^{x^2}$
   b. $f(x) = \tan(\log_5 x)$ $\Rightarrow f'(x)=\dfrac{\sec^2(\log_5 x)}{x\ln 5}$
   c. $f(x) = x^{\sin x}$ $\quad \left[\cos(x)\ln x+\frac{\sin x}{x}\right]x^{\sin x}$
   d. $f(x) = \sinh(3x)$ $\Rightarrow f'=3\cosh(3x)$
   e. $f(x) = \dfrac{\cosh x}{x}$ $\Rightarrow$ $\dfrac{\sinh(x)}{x}-\dfrac{\cosh(x)}{x^2}$

Math 1432 - Final Review

1. a. Find tangent line at $x=0$

$f(x) = \ln(6x+1) + e^{2x}$.

$f'(x) = \frac{6}{6x+1} + 2e^{2x}$. $\Rightarrow$ slope @ $x=0$: $f'(0) = 6+2\cdot e^0 = 6+2\cdot 1 = 8$

point $\Rightarrow (0, f(0)) = (0, \ln 1 + e^0) = (0, 0+1) = (0,1)$

$\Rightarrow$ tangent line : $\boxed{y-1 = 8x}$

b. $f(x) = \ln(2x+1) - 3e^{-4x}$

$f'(x) = \frac{2}{2x+1} + 12e^{-4x} \Rightarrow$ slope @ $x=0$: $f'(0) = +2 + 12\,e^0 = +2+12$
$= 14$.

point $(0, f(0)) = (0, \ln 1 - 3e^0) = (0, -3)$

$\Rightarrow$ tangent line : $y+3 = 14x$

c. $f(x) = \sqrt{9-x^2} = (9-x^2)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(9-x^2)^{-\frac{1}{2}} \cdot (-2x) \Rightarrow$ slope @ $x=0$: $f'(0) = 0$.
$= -\frac{x}{(9-x^2)^{\frac{1}{2}}}$

point $(0, f(0)) = (0, 3)$

tangent line : $y - 3 = 0$

2. Find inverse of f

a. $f(x)=\frac{2}{3-x}$

① let $y=f(x)=\frac{2}{3-x}$

② Switching x and y $\Rightarrow$ $x=\frac{2}{3-y}$.

③ Solving y: $\Rightarrow 3-y=\frac{2}{x} \Rightarrow \boxed{3-\frac{2}{x}=y}$

b. $f(x)=\frac{x+1}{x+2}$.

① let $y=f(x)=\frac{x+1}{x+2}=\frac{x+1+1-1}{x+2}=\frac{x+2-1}{x+2}=1-\frac{1}{x+2}$.

② Switching x and y: $x=1-\frac{1}{y+2}$.

③ Solving y $\Rightarrow \frac{1}{y+2}=1-x \Rightarrow y+2=\frac{1}{1-x} \Rightarrow \boxed{y=\frac{1}{1-x}-2}$

3. Find derivative of f^{-1}: $f(x)$ is invertiable and differentiable

$\Rightarrow$ If $f(a)=b$, then $(f^{-1})'(b)=\dfrac{1}{f'(a)}$

a. $f(x)=x^3+1$, $f(\underset{a}{2})=\underset{b}{9}$.

$f'(x)=3x^2 \Rightarrow$ increasing $\Rightarrow$ one-to-one $\Rightarrow f^{-1}$ exists.

$$\Rightarrow (f^{-1})'(9)=\frac{1}{f'(2)}=\boxed{\frac{1}{12}}.$$

b. $f(\overset{a}{-3})=\overset{b}{1}$, $f(1)=2$, $f'(\overset{a}{-3})=3$, $f'(1)=-2$

$$(f^{-1})'(\underset{b}{1})=\frac{1}{f'(-3)}=\boxed{\frac{1}{3}}$$

c. $f(x)$ passes through $(3,-2) \Rightarrow f(\overset{a}{3})=-2 \leftarrow b$

$(-2,1) \Rightarrow f(-2)=1$.

The slope at $x=3$ is $-\frac{1}{4} \Rightarrow f'(3)=-\frac{1}{4}$

Find $(f^{-1})'(\underset{b}{-2})=\dfrac{1}{f'(3)}=\dfrac{1}{-\frac{1}{4}}=\boxed{-4}$

4. Tangent line and normal line at given point

a. $x(t) = -2\cos 2t$, $y(t) = 4+2t$, @ $(-2, \overset{4+2t}{\overset{\shortparallel}{4}})$. $\Rightarrow t=0$

$(4+2t=4 \Rightarrow t=0)$

Tangent: $\left.\dfrac{dy}{dx}\right|_{t=0} = \left.\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=0} = \left.\dfrac{2}{4\sin 2t}\right|_{t=0} \Rightarrow$ Vertical line $\Rightarrow X = -2$.

$\Rightarrow$ normal line : $y = 4$.

b. $x(t) = 3\cos(3t) + 2t$, $y(t) = 1+5t$ @ $(3, \overset{1+5t}{\overset{\shortparallel}{1}}) \Rightarrow 1+5t = 1 \Rightarrow t=0$.

Tangent: $\left.\dfrac{dy}{dx}\right|_{t=0} = \left.\dfrac{5}{-9\sin(3t)+2}\right|_{t=0} = \dfrac{5}{2}$. $\Rightarrow$ tangent line $y-1 = \frac{5}{2}(x-3)$

Slope of tangent line

Slope of normal line : $S_n \times S_T = -1$.

$\Rightarrow S_n = \dfrac{-1}{S_T} = -\dfrac{2}{5}$ $\Rightarrow$ normal line $y-1 = -\frac{2}{5}(X-3)$

5. Give a x, y equation

a. $x(t) = -1 + 3\cos t$, $y(t) = 1 + 2\sin t$

$\Rightarrow \dfrac{x+1}{3} = \cos t$, $\dfrac{y-1}{2} = \sin t$

$\Rightarrow \underline{1}^2 = (\cos t)^2 + (\sin t)^2 = \underline{\left(\dfrac{x+1}{3}\right)^2 + \left(\dfrac{y-1}{2}\right)^2}$

b. $x(t) = -1 + 3\cosh(t)$, $y(t) = 1 + 2\sinh(t)$

$\Rightarrow \dfrac{x+1}{3} = \cosh(t)$, $\dfrac{y-1}{2} = \sinh(t)$.

$\Rightarrow \underline{1}^2 = [\cosh(t)]^2 - [\sinh(t)]^2 = \underline{\left(\dfrac{x+1}{3}\right)^2 - \left(\dfrac{y-1}{2}\right)^2}$

c. $x(t) = -1 + 4e^t$, $y(t) = 2 + 3e^{-t}$

$\Rightarrow \dfrac{x+1}{4} = e^t$, $\dfrac{y-2}{3} = e^{-t} = \dfrac{1}{e^t}$

$\Rightarrow \dfrac{x+1}{4} = \dfrac{3}{y-2}$

6. a. $f(x) = 3^{x^2}$

$\ln f(x) = x^2 \cdot \ln 3$

$\xrightarrow{\frac{d}{dx}} \frac{f'(x)}{f(x)} = 2x\ln 3 \Rightarrow \boxed{f'(x) = [2x\ln 3]\, 3^{x^2}}$

b. $f(x) = \tan(\log_5 x) = \tan\left(\frac{\ln x}{\ln 5}\right)$

$f'(x) = [\sec^2(\log_5 x)] \cdot \frac{1}{x \cdot \ln 5}$

c. $f(x) = x^{\sin(x)}$

$\ln f(x) = \sin(x) \ln x$

$\xrightarrow{\frac{d}{dx}} \frac{f'(x)}{f(x)} = \cos(x)\ln x + \sin(x) \cdot \frac{1}{x}$ $\cancel{\Rightarrow f'(x)}$

$\Rightarrow f'(x) = \left[\cos(x)\ln x + \frac{\sin(x)}{x}\right] \cdot x^{\sin(x)}$

d. $f(x) = \sinh(3x)$

$f'(x) = 3\cosh(3x)$

e. $f(x) = \frac{\cosh(x)}{x} = [\cosh(x)] \cdot x^{-1}$

$f'(x) = \frac{\sinh(x)}{x} + \frac{-[\cosh(x)]}{x^2}$

7. Integrate:

   a. $\int(\cosh(3x)+\sinh(2x))dx$

   b. $\int 4^{3x}\,dx$

   c. $\int\frac{\log_2\left(x^3\right)}{x}dx$

   d. $\int\left(2^{7x}-\sinh(5x)\right)dx$

   e. $\int\frac{\sin(3x)}{16+\cos^2(3x)}dx$

   f. $\int\frac{6x}{4+x^4}dx$

   g. $\int\tan(3x)dx$

   h. $\int\frac{\arctan(3x)}{1+9x^2}dx$

   i. $\int\frac{1}{\sqrt{4+x^2}}dx$

   j. $\int\sqrt{9-x^2}\,dx$

   k. $\int 3\ln(4x)dx$

   l. $\int x^2e^x\,dx$

   m. $\int\frac{5x+14}{(x+1)(x^2-4)}dx$

   n. $\int\frac{x^2+5x+2}{(x+1)(x^2+1)}dx$

   o. $\int\frac{2x^2}{\sqrt{9-x^2}}dx$

   p. $\int 2\arctan(10x)dx$

   q. $\int 3x\cos(2x)dx$

   r. $\int\frac{4}{\left(\sqrt{25-x^2}\right)^3}dx$

   s. $\int\frac{5}{x^2\sqrt{36-x^2}}dx$

8. Write an expression for the nth term of the sequence:

   a. 1, 4, 7, 10, ...

   b. 2, -1, $\frac{1}{2}$, $-\frac{1}{4}$, $\frac{1}{8}$, ...

7.

a. $\int(\cosh(3x)+\sinh(2x))\,dx = \dfrac{\sinh(3x)}{3}+\dfrac{\cosh(2x)}{2}+C$

b. $\int 4^{3x}\,dx = \dfrac{4^{3x}}{3\ln 4}+C$

c. $\int \dfrac{\log_2(x^3)}{x}\,dx = \dfrac{1}{\ln 2}\int \dfrac{\ln x^3}{x}\,dx = \dfrac{3}{\ln 2}\int\dfrac{\ln x}{x}\,dx = \dfrac{3}{\ln 2}\cdot\dfrac{(\ln x)^2}{2}+C$

d. $\int(2^{7x}-\sinh(5x))\,dx = \dfrac{2^{7x}}{7\ln 2}-\dfrac{\cosh(5x)}{5}+C$

e. $\int \dfrac{\sin(3x)}{16+\cos^2(3x)}\,dx = -\dfrac{1}{3}\int\dfrac{du}{16+u^2} = -\dfrac{1}{3}\cdot\dfrac{1}{4}\arctan\left(\dfrac{u}{4}\right)+C$

$\boxed{\text{let } \cos(3x)=u \Rightarrow du=-3\sin(3x)dx}$

$= -\dfrac{1}{12}\arctan\left(\dfrac{\cos(3x)}{4}\right)+C$

f. $\int \dfrac{6x^{=3\cdot 2x}}{4+x^4}\,dx = 3\int\dfrac{du}{4+u^2} = 3\cdot\dfrac{1}{2}\arctan\left(\dfrac{u}{2}\right)+C = \dfrac{3}{2}\arctan\left(\dfrac{x^2}{2}\right)+C$

$\boxed{\text{let } u=x^2,\ du=2x\,dx}$

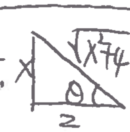
g. $\int\tan(3x)\,dx = \dfrac{1}{3}\ln|\sec(3x)|+C$

h. $\int\dfrac{\arctan(3x)}{1+9x^2}\,dx = \dfrac{1}{3}\int u\,du = \dfrac{1}{3}\cdot\dfrac{u^2}{2}+C = \dfrac{1}{6}[\arctan(3x)]^2+C$

$\boxed{\text{let } u=\arctan(3x),\ du=\dfrac{dx}{1+(3x)^2}\cdot 3=\dfrac{3dx}{1+9x^2}}$

i. $\int\dfrac{1}{\sqrt{4+x^2}}\,dx = \int\dfrac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C$

let $x=2\tan\theta$, $dx=2\sec^2\theta\,d\theta$ $\Rightarrow \sqrt{4+x^2}=\sqrt{4+4\tan^2\theta}=\sqrt{4\sec^2\theta}=2\sec\theta$ $\Rightarrow \tan\theta=\dfrac{x}{2}$,

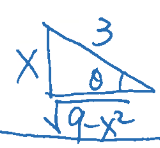


$= \ln\left|\dfrac{\sqrt{x^2+4}}{2}+\dfrac{x}{2}\right|+C$

j. $\int\sqrt{9-x^2}\,dx = \int 3\cos\theta\cdot 3\cos\theta\,d\theta = 9\int\cos^2\theta\,d\theta = 9\int\left[\dfrac{1}{2}+\dfrac{1}{2}\cos 2\theta\right]d\theta$

let $x=3\sin\theta$, $dx=3\cos\theta\,d\theta$ $\Rightarrow \sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=3\cos\theta$

$\Rightarrow \sin\theta=\dfrac{x}{3}$,



$= \dfrac{9}{2}\theta+\dfrac{9}{4}\sin 2\theta+C$

$= \dfrac{9}{2}\arcsin\left(\dfrac{x}{3}\right)+\dfrac{9}{4}\cdot\dfrac{x}{3}+C$

7.

k. $\int \underset{A}{\underline{3}}\,\underset{T}{\underline{\ln(4x)}}\,dx = 3x\ln(4x) - \int 3\,dx = 3x\ln(4x) - 3x + C$

$u = \ln(4x)$ $\quad dv = 3dx$

$du = \frac{4\,dx}{4x}$ $\quad V = 3x$

u	dv	
x^2	e^x	+
$2x$	e^x	−
2	e^x	+
0	e^x	−
		+

l. $\int \underset{A}{\underline{x^2}}\,\underset{E}{\underline{e^x}}\,dx = x^2e^x - 2x\cdot e^x + 2e^x + C$

m. $\int \frac{5x+14}{(x+1)(x^2-4)}dx = \int\left[\frac{-3}{x+1} + \frac{1}{x+2} + \frac{2}{x-2}\right]dx$

$= -3\ln|x+1| + \ln|x+2| + 2\ln|x-2| + C$

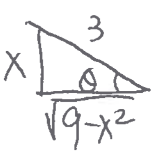
n. $\int \frac{x^2+5x+2}{(x+1)(x^2+1)}dx = \int\left[\frac{-1}{x+1} + \frac{2x+3}{x^2+1}\right]dx$

$= \int\left[\frac{-1}{x+1} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\right]dx = -\ln|x+1| + \ln|x^2+1| + 3\arctan(x) + C$

o. $\int \frac{2x^2}{\sqrt{9-x^2}}dx = \int \frac{2\cdot(3\sin\theta)^2\cdot 3\cos\theta\, d\theta}{3\cos\theta} = 18\int \sin^2\theta\, d\theta$

Let $x = 3\sin\theta$.
$dx = 3\cos\theta\, d\theta$

$\Rightarrow \sin\theta = \frac{x}{3}$



$= 18\int\left[\frac{1}{2} - \frac{1}{2}\cos 2\theta\right]d\theta = 18\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C$

$= 9\theta - \frac{9}{2}\sin 2\theta + C$

$= 9\theta - \frac{9}{2}\cdot 2\cos\theta\sin\theta + C$

$= 9\theta - 9\cos\theta\cdot\sin\theta + C$

$= 9\arcsin\left(\frac{x}{3}\right) - 9\cdot\frac{\sqrt{9-x^2}}{3}\cdot\frac{x}{3} + C$

$= 9\arcsin\left(\frac{x}{3}\right) - x\sqrt{9-x^2} + C$

p. $\int \underset{A}{\underline{2}}\,\underset{I}{\underline{\arctan(10x)}}\,dx = 2x\arctan(10x) - \int \frac{20x}{1+100x^2}dx$

$u = \arctan(10x)$ $\quad dv = 2dx$

$du = \frac{10\,dx}{1+(10x)^2}$ $\quad V = 2x$

$= 2x\cdot\arctan(10x) - \frac{1}{100}\ln|1+100x^2| + C$

7.

a. $\int \underbrace{3x}_{A}\ \underbrace{\cos(2x)}_{T}\,dx = \frac{3}{2}x\sin(2x) + \frac{3}{4}\cos(2x) + C$

u	dV	
$3x$	$\cos(2x)$	$+$
3	$\frac{\sin(2x)}{2}$	$-$
0	$-\frac{\cos(2x)}{4}$	$+$
		$-$

8. Find an expression for n-th term

a. $\underbrace{1, 4}_{3}, \underbrace{4, 7}_{3}, \underbrace{7, 10}_{3}, \ldots \Rightarrow 3$ is common difference.

$\underbrace{\boxed{?}\ \ 1}_{3} \Rightarrow ? = 1-3 = -2. \Rightarrow a_n = -2+3n,\ n \in \mathbb{N}.$

b. $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8} \cdots \Rightarrow (-1)^2 2^1, (-1)^3 2^0, (-1)^4 2^{-1}, (-1)^5 2^{-2} \ldots$

alternating $\Rightarrow (-1)^{n+1} \Rightarrow a_n = (-1)^{n+1}\, 2^{2-n},\ n \in \mathbb{N}$

9. a. $a_n - a_{n+1} = \frac{2n}{1+n} - \frac{2(n+1)}{1+(n+1)} = \frac{2n}{1+n} - \frac{2n+2}{2+n} = \frac{(2n)(2+n)-(2n+2)(1+n)}{(1+n)(2+n)} = \frac{-2}{(1+n)(2+n)} < 0$

$\Leftrightarrow a_n - a_{n+1} < 0 \Leftrightarrow a_n < a_{n+1}$ which means a_n is $\boxed{\text{increasing}}$.

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{1+n} \xrightarrow{\text{leading coefficient}} 2$ and a_n is increasing, then $\boxed{a_1 \le a_n < 2}$, $a_1 = \frac{2}{3}$ glb, 2 lub

9. Determine if the following sequences are monotonic. Also indicate if the sequence is bounded and if it is give the least upper bound and/or greatest lower bound.

a. $a_n = \dfrac{2n}{1+n}$

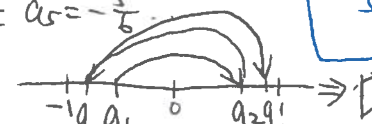
b. $a_n = \dfrac{\cos n}{n}$

b. Since "cos(n)" isn't monotonic, a_n isn't monotonic.
But since $-\frac{1}{n} \le \frac{\cos(n)}{n} \le \frac{1}{n}$, we known a_n is bounded.
check $\frac{\cos(1)}{1} = 0.54$, $\frac{\cos(2)}{2} = -0.20$, $\frac{\cos(3)}{3} = -0.32$, $\frac{\cos(4)}{4} = -0.16$, $\frac{\cos(5)}{5} = 0.05$, $\vdots$
we have lub and glb

10. Determine if the following sequences converge or diverge. If they converge, give the limit.

$a_n = (-1)^n\left(\frac{n}{n+1}\right) \to a_1 = -\frac{1}{2}$, $a_2 = \frac{2}{3}$, $a_3 = -\frac{3}{4}$, $a_4 = \frac{4}{5}$, $a_5 = -\frac{5}{6}$

Diverge a. $\left\{(-1)^n\left(\dfrac{n}{n+1}\right)\right\}$ See this sequence on number line:



$\Rightarrow$ Div.

Converge b. $\left\{\dfrac{6n^2-2n+1}{4n^2-1}\right\}$ $\deg(P) = \deg(Q)$. $\left\{\frac{6n^2-2n+1}{4n^2-1}\right\} \xrightarrow{\text{leading coefficient}} \frac{3}{2}$ as $n \to \infty$

Diverge. c. $\left\{\dfrac{(n+2)!}{n!}\right\} = \{(n+2)(n+1)\} \to \infty$ as $n \to \infty$ Diverge

Converge d. $\left\{\dfrac{3}{e^n}\right\}$ Top is fixed and the bottom tends to ∞ as $n\to\infty$, $\left\{\frac{3}{e^n}\right\} \to 0$ as $n \to \infty$

Converge e. $\left\{\dfrac{4n+1}{n^2-3n}\right\}$ $\deg p < \deg(Q)$ $\left\{\frac{4n+1}{n^2-3n}\right\} \to 0$ as $n \to \infty$

Converge. f. $\left\{\dfrac{e^n}{n^3}\right\}$ n^3 is faster than e^n ($e = 2.71\cdots$), $\left\{\frac{e^n}{n^3}\right\} \to 0$ as $n \to \infty$

11. Determine if the following series (A) converge absolutely, (B) converge conditionally or (C) diverge.

(B) a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$

(A) b. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$

(B) c. $\displaystyle\sum_{n=0}^{\infty} \frac{4n(-1)^n}{3n^2+2n+1}$

(B) d. $\displaystyle\sum_{n=0}^{\infty} \frac{3(-1)^n}{\sqrt{3n^2+2n+1}}$

(C) e. $\displaystyle\sum_{n=0}^{\infty} \frac{3n(-1)^n}{\sqrt{3n^2+2n+1}}$

(C) f. $\displaystyle\sum_{n=0}^{\infty} \left(4(-1)^n\left(\frac{n}{n+3}\right)^n\right)$

11.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$.

X (A) $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}\sqrt{n}}{n+3} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3} \sim \sum \frac{n^{\frac{1}{2}}}{n} = \sum \frac{1}{n^{\frac{1}{2}}}$ Diverges $\Rightarrow$ NOT A.C.

V (B) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$, let $b_n = \frac{\sqrt{n}}{n+3}$, $0 \le b_{n+1} < b_n$ and $b_n \to 0$ as $n \to \infty$

by Alternating series Test, it Converges $\Rightarrow$ conditionally converges.

b. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ ($\cos \pi n = (-1)^n$. Check it!)

V (A) $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2}$ converges by p-series.

c. $\sum_{n=0}^{\infty} \frac{(-1)^n 4n}{3n^2+2n+1}$

X (A) $\sum_{n=0}^{\infty} \left| \frac{(-1)^n 4n}{3n^2+2n+1} \right| = \sum_{n=0}^{\infty} \frac{4n}{3n^2+2n+1} \sim \sum_{n=0}^{\infty} \frac{n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ NOT A.C.

✓ (B) $\sum_{n=0}^{\infty} (-1)^n \frac{4n}{3n^2+2n+1}$. let $b_n = \frac{4n}{3n^2+2n+1}$ $\Rightarrow 0 \le b_{n+1} < b_n$ and $b_n \to 0$

by Alternating Series test it converges

d. $\sum_{n=0}^{\infty} (-1)^n \frac{3}{\sqrt{3n^2+2n+1}}$

X (A) $\sum_{n=0}^{\infty} \left| (-1)^n \frac{3}{\sqrt{3n^2+2n+1}} \right| = \sum_{n=0}^{\infty} \frac{3}{\sqrt{3n^2+2n+1}} \sim \sum \frac{1}{\sqrt{n^2}} = \sum \frac{1}{n}$ diverges $\Rightarrow$ NOT A.C.

V (B) $\sum_{n=0}^{\infty} (-1)^n \frac{3}{\sqrt{3n^2+2n+1}}$ let $b_n = \frac{3}{\sqrt{3n^2+2n+1}}$ $\Rightarrow 0 \le b_{n+1} < b_n$ and $b_n \to 0$.

by Alternating Series, it converges

11.

e. $\sum_{n=0}^{\infty} \frac{(-1)^n 3n}{\sqrt{3n^2+2n+1}}$

X (A) $\sum_{n=0}^{\infty} \left| \frac{(-1)^n 3n}{\sqrt{3n^2+2n+1}} \right| = \sum_{n=0}^{\infty} \frac{3n}{\sqrt{3n^2+2n+1}} \sim \sum \frac{n}{\sqrt{n^2}} = \sum \frac{n}{n} = \sum 1$ diverges (by B.D.T.)

X (B) $\sum_{n=0}^{\infty} (-1)^n \frac{3n}{\sqrt{3n^2+2n+1}}$ let $a_n = (-1)^n \frac{3n}{\sqrt{3n^2+2n+1}}$, $a_n \not\to 0$ as $n \to \infty$ diverges (by B.D.T.).

√ (C) Divergee.

f. $\sum_{n=0}^{\infty} \left(4(-1)^n \cdot \left(\frac{n}{n+3}\right)^n\right)$

since $\left(\frac{n}{n+3}\right)^n = \left(\frac{n+3}{n}\right)^{-n} = \left(1+\frac{3}{n}\right)^{-n} = \left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}\cdot(-3)}$

$= \left[\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}}\right]^{-3} \to e^{-3} \neq 0 \Rightarrow$ Divergee $\Rightarrow$ (C).

g. $\sum_{n=0}^{\infty} \left(\frac{(-1)^n 2\arctan(n)}{n^3+n^2+3}\right)$

$|\arctan(n)| < \frac{\pi}{2}$

√ (A) $\sum_{n=0}^{\infty} \left| \frac{(-1)^n 2\arctan(n)}{n^3+n^2+3} \right| = \sum_{n=0}^{\infty} \frac{2|\arctan(n)|}{n^3+n^2+3} \leq \sum_{n=0}^{\infty} \frac{2\cdot\frac{\pi}{2}}{n^3+n^2+3} \sim \sum \frac{1}{n^3}$ converges

h. $\sum_{n=0}^{\infty} \left(\frac{(-1)^n 3^n}{4^n+3n}\right)$

← Geometric series

√ (A) $\sum_{n=0}^{\infty} \left| \frac{(-1)^n 3^n}{4^n+3n} \right| = \sum_{n=0}^{\infty} \frac{3^n}{4^n+3n} \leq \sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges

11.

i $\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3}{(n+2)\ln(n+2)}\right)$

X (A) $\sum_{n=0}^{\infty}\left|\frac{(-1)^n 3}{(n+2)\ln(n+2)}\right| = \sum_{n=0}^{\infty}\frac{3}{(n+2)\ln(n+2)}$

Check convergence by integral test.

$\int_0^{\infty}\frac{3}{(x+2)\ln(x+2)}dx = 3\ln(\ln(x+2))\Big|_0^{\infty} \to$ Diverges

V (B) $\sum_{n=0}^{\infty}\frac{(-1)^n 3}{(n+2)\ln(n+2)}$, let $b_n=\frac{3}{(n+2)(\ln(n+2))} \Rightarrow 0\le b_{n+1}<b_n$ and $b_n\to 0$.

By Alternating series test, it converges.

12、

Geometric series. $a=2$, $r=-\frac{4}{9}$

a. $\sum_{n=0}^{\infty}2\left(-\frac{4}{9}\right)^n \overset{\downarrow}{=} \frac{a}{1-r} = \frac{2}{1-(-\frac{4}{9})} = 2\cdot\frac{9}{13} = \boxed{\frac{18}{13}}$

both sum exist by Geometric

b. $\sum_{n=0}^{\infty}\left(\frac{1}{3^n}-\frac{5}{6^n}\right) \overset{\downarrow}{=} \sum_{n=0}^{\infty}\frac{1}{3^n} - \sum_{n=0}^{\infty}5\frac{1}{6^n} = \frac{1}{1-\frac{1}{3}} - \frac{5}{1-\frac{1}{6}}$

$a=1, r=\frac{1}{3}$ $\quad$ $a=5, r=\frac{1}{6}$

$= \frac{3}{2} - 5\cdot\frac{6}{5} = \boxed{-\frac{9}{2}}$

(A) g. $\sum_{n=0}^{\infty}\left(\frac{2(-1)^n \arctan n}{3+n^2+n^3}\right)$

(A) h. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3^n}{4^n+3n}\right)$

(B) i. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3}{(n+2)\ln(n+2)}\right)$

12. Find the sum of the following convergent series:

a. $\sum_{n=0}^{\infty} 2\left(-\frac{4}{9}\right)^n = \frac{18}{13}$

b. $\sum_{n=0}^{\infty}\left(\frac{1}{3^n}-\frac{5}{6^n}\right) = -\frac{9}{2}$

13. State the indeterminate form and compute the following limits :

$\left(\frac{\infty}{\infty}\right)$, 0 a. $\lim_{n\to\infty}\frac{\ln(n+4)}{n+2} \overset{(L')}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty}\frac{\frac{1}{n+4}}{1} = \lim_{n\to\infty}\frac{1}{n+4} = 0$

(∞^0), 1 b. $\lim_{n\to\infty}(3n)^{\frac{2}{n}}$ ⇒ Take ln ⇒ $\lim_{n\to\infty}\ln(3n)^{\frac{2}{n}} = \lim_{n\to\infty}\frac{2}{n}\ln(3n) \overset{(L')}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty}\frac{\frac{2}{3n}}{1} = 0$

Then taking e back, we have $\lim_{n\to\infty}(3n)^{\frac{2}{n}} = e^0 = 1$

(1^∞), e^6 c. $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{(\frac{n}{3})}\right)^{\frac{n}{3}}\right]^6 = \left[\lim_{n\to\infty}\left(1+\frac{1}{(\frac{n}{3})}\right)^{\frac{n}{3}}\right]^6 = e^6$

$\left(\frac{0}{0}\right)$, $-\frac{1}{3}$ d. $\lim_{x\to 0}\frac{x-\sin(2x)}{x+\sin(2x)} \overset{L'}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{1-2\cos(2x)}{1+2\cos(2x)} = \frac{-1}{3}$

$\left(\frac{0}{0}\right)$, $\frac{1}{2}$ e. $\lim_{x\to 0}\frac{e^{x^2}-1}{2x^2} \overset{(L')}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{2xe^{x^2}}{4x} \overset{(L')}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{2e^{x^2}+4x^2e^{x^2}}{4} = \frac{2}{4} = \frac{1}{2}$

(∞^0), 1, f. $\lim_{x\to 0+}\left(\frac{1}{x}\right)^x$ ⇒ Take "ln" We have $\lim_{x\to 0^+}\ln\left(\frac{1}{x}\right)^x = \lim_{x\to 0^+} x\ln\left(\frac{1}{x}\right) = \lim_{x\to 0^+}\frac{\ln(\frac{1}{x})}{\frac{1}{x}} \overset{(L')}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{x\to 0^+}\frac{x\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = 0$ ⇒ Take "e" $\lim_{x\to 0^+}\left(\frac{1}{x}\right)^x = e^0 = 1$

$\left(\frac{0}{0}\right)$, $\frac{1}{6}$ g. $\lim_{x\to 0}\frac{3e^{x/3}-(3+x)}{x^2} \overset{(L')}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{e^{\frac{x}{3}}-1}{2x} \overset{(L')}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{\frac{1}{3}e^{\frac{x}{3}}}{2} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

$\left(\frac{\infty}{\infty}\right)$ Div. h. $\lim_{x\to\infty}\frac{x^2}{\ln x} \overset{(L')}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{x\to\infty}\frac{2x}{\frac{1}{x}} = \infty$ Div

$\left(\frac{0}{0}\right)$, $-\frac{1}{2}$ ✓i. $\lim_{x\to 0}\frac{1+x-e^x}{x(e^x-1)} \overset{(L')}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{1-e^x}{xe^x+e^x-1} \overset{(L')}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{-e^x}{xe^x+e^x+e^x} = \frac{-1}{2}$

$\left(\frac{0}{0}\right)$, 4 j. $\lim_{x\to 0}\frac{\arctan(4x)}{x} \overset{(L')}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0}\frac{\frac{1}{1+(4x)^2}\cdot 4}{1} = 4$

✓ k. $\lim_{x\to 0}\frac{4e^{(\frac{x}{4})}-(4+x)}{x^2}$

✓ l. $\lim_{x\to 0}\frac{x-\ln(x+1)}{1-\cos(3x)}$

14 a. $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n^2+2} \xrightarrow[\text{derivative}]{\left(\frac{d}{dx}\right)} \boxed{\sum_{n=1}^{\infty} \frac{n(n+1)x^{n-1}}{n^2+2}}$

b. $\sum_{n=0}^{\infty} \frac{x^n}{2n+1} = \frac{1}{1} + \sum_{n=1}^{\infty} \frac{x^n}{2n+1} \xrightarrow{\frac{d}{dx}} \boxed{\sum_{n=1}^{\infty} \frac{n x^{n-1}}{2n+1}}$

14. Give the derivative of each power series below:

a. $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n^2+2}$ $\xrightarrow[\text{antider.}]{} F(x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n^2+2} \xrightarrow{F(0)=0} C=0. \Rightarrow \boxed{F(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+2}}$

b. $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$ $\xrightarrow[\text{antider.}]{} F(x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\cdot(2n+1)} \xrightarrow{F(0)=0} C=0 \Rightarrow \boxed{F(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(2n+1)}}$

15. For each of the problems in number 14, give the antiderivate F of the power series so that F(0)=0.

16. Evaluate each improper integral:

$= \int_0^{27} \frac{dx}{x^{\frac{2}{3}}} = \lim_{a \to 0} \left[\int_a^{27} x^{-\frac{2}{3}} dx \right] = \lim_{a \to 0} \left[3x^{\frac{1}{3}} \right]_a^{27}$

$= \lim_{a \to 0} \left[3 \cdot 27^{\frac{1}{3}} - 3 \cdot a^{\frac{1}{3}} \right] = 3 \cdot 3 = \boxed{9}$

9 √ a. $\int_0^{27} x^{-2/3} dx$

4 √ b. $\int_0^4 \frac{dx}{\sqrt{4-x}}$ $\rightarrow \lim_{b \to 4} \left[\int_0^b \frac{dx}{\sqrt{4-x}} \right] = \lim_{b \to 4} \left[-2(4-x)^{\frac{1}{2}} \right]_0^b = \lim_{b \to 4} \left[-2(4-b)^{\frac{1}{2}} + 2(4-0)^{\frac{1}{2}} \right]$

$(4-x)^{-\frac{1}{2}}$ $\quad |a|<|b| \Rightarrow$ loop $\quad = \boxed{4}$

17. Find the formula for the area of $r = 1 + 2\sin\theta$
   a. Inside inner loop
   b. Inside outer loop but outside inner loop
   c. Inside outer loop and below x-axis

18. Find the smallest value of n so that the nth degree Taylor Polynomial for $f(x) = \ln(1+x)$ centered at x = 0 approximates $\ln(2)$ with an error of no more than 0.001 (also be able to do this with some of the other Taylor Polynomials)

19. Find the radius of convergence and interval of convergence for the following Power series:

a. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)3^{n+1}}$

e. $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2^k (k^2+1)}$

b. $\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n$

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$

d. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n n!}{n^n}$

20. Use logarithmic differentiation to find the derivative of:
   a. $y = (3x-1)^{\sin(x)}$
   b. $y = (x+1)^{\ln(x)}$
   c. $y = (x^2+2)^{(\frac{1}{\ln x})}$

17. $r = 1 + 2\sin\theta$, $|a| = 1 < 2 = |b| \Rightarrow$ loop

$[r, \theta] \Rightarrow [1, 0]$

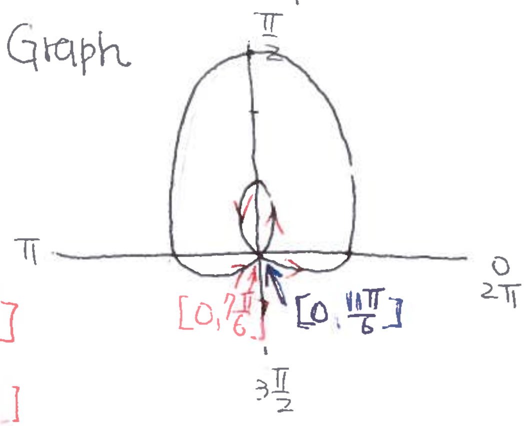
$[3, \frac{\pi}{2}]$

$[1, \pi]$

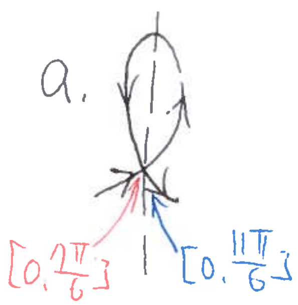
$[-1, \frac{3\pi}{2}]$ $[0, \frac{7\pi}{6}]$

$[1, 2\pi]$ $[0, \frac{11\pi}{6}]$

Graph

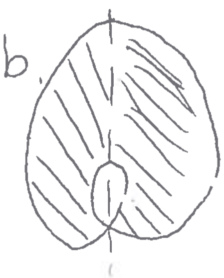


a.

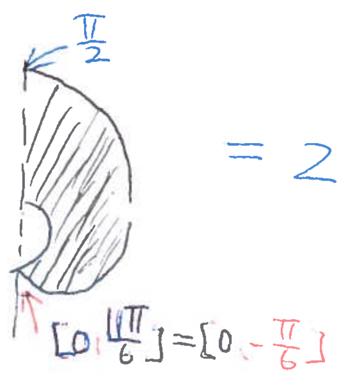


$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2} r^2 \, d\theta = \boxed{\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1+2\sin\theta)^2 \, d\theta}$$

b.

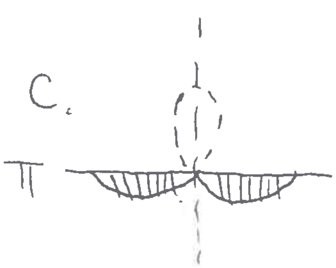


$= 2\times$

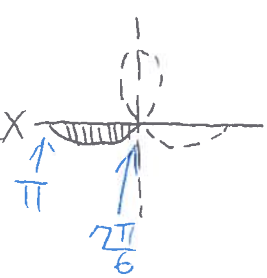


$$= 2 \times \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (1+2\sin\theta)^2 \, d\theta$$

c.



$= 2\times$



$$= 2 \times \frac{1}{2} \int_{\pi}^{\frac{7\pi}{6}} (1+2\sin\theta)^2 \, d\theta.$$

18. Given $f(x)=\ln(1+x)$. $R_n = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$

To Find n s.t. $R_n(1) \leq 0.001$

① find $f^{(n+1)}(x)$. $f'(x)=(1+x)^{-1}$, $f''(x)=-1(1+x)^{-2}$, $f'''(x)=(-1)(-2)(1+x)^{-3}$

$f^{(4)}(x)=(-1)(-2)(-3)(1+x)^{-4}$, ... $f^{(n+1)}(x)=(-1)(-2)\cdots(-n)(1+x)^{-(n+1)}$

$|f^{(n+1)}(c)| \leq n!$ $\quad = (-1)^n \cdot n! \, (1+x)^{-n-1}$

② $|R_n(1)| = \left| \frac{(-1)^n \cdot n! \, (1+c)^{-n-1}}{(n+1)!} \cdot (1)^{n+1} \right| \leq \left| \frac{n!}{(n+1)!} \right| = \frac{1}{n+1} \leq 0.001$

$\Rightarrow \frac{1}{n+1} \leq \frac{1}{1000} \Rightarrow n = \boxed{999}$

19. Find radius of convergence and interval of convergence.

a. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)\,3^{n+1}}$

let $a_n = \frac{(x-2)^{n+1}}{[n+1]\,3^{n+1}}$, By Ratio test, If this series converges, we have

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-2)^{(n+1)+1}}{[(n+1)+1]\,3^{(n+1)+1}} \cdot \frac{(n+1)3^{n+1}}{(x-2)^{n+1}} \right|$

$= \left| \frac{(x-2)^{n+2}}{(n+2)\,3^{n+2}} \cdot \frac{(n+1)\,3^{n+1}}{(x-2)^{n+1}} \right| = \left| \frac{n+1}{n+2} \cdot \frac{1}{3} \cdot (x-2) \right| \to \left| \frac{x-2}{3} \right| < 1$

$\Rightarrow \left| \frac{x-2}{3} \right| < 1 \Rightarrow |x-2| < \underset{\uparrow \text{ radius}}{3} \Rightarrow -3 < x-2 < 3 \Rightarrow -1 < x < 5$

interval

check $x=-1$, $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{(n+1)\,3^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$ converges (Alternating)

check $x=5$. $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(n+1)3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges (p-series)

$\Rightarrow$ $-1 < x \leq 5$ or $x \in (-1, 5]$

19.

b. $\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n$

let $a_n = \frac{1}{3^n}(x-1)^n$. By Ratio test. If this series converges, we have

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x-1)^n}\right| = \left|\frac{x-1}{3}\right| < 1$

$\Rightarrow \left|\frac{x-1}{3}\right| < 1 \Rightarrow |x-1| < 3 \Rightarrow -3 < x-1 < 3 \Rightarrow -2 < x < 4$

radius ↑ (at 3) interval

check $x=4$, $\sum_{n=0}^{\infty} \frac{1}{3^n}\cdot(4-1)^n = \sum \frac{3^n}{3^n} = \sum 1$ diverges ($1 \not\to 0$. by BDT). $\Rightarrow -2 < x < 4$

check $x=-2$ $\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ diverges ($1 \not\to 0$ by BDT)

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$

let $a_n = \frac{(-1)^{n+1} x^n}{4^n}$. By Ratio test, if this series converges, we have

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2} x^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^{n+1} x^n}\right| = \left|(-1)\frac{x}{4}\right| \to \left|\frac{x}{4}\right| < 1$

$\Rightarrow \left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4 \Rightarrow -4 < x < 4$

radius ↑ (at 4) ~~Interval~~

Check $x=4$, $\sum \frac{(-1)^{n+1} 4^n}{4^n} = \sum (-1)^{n+1}$ diverges ($\pm 1 \not\to 0$ B.D.T).

check $x=-4$, $\sum \frac{(-1)^{n+1}(-4)^n}{4^n} = \sum (-1)^{2n+1} = \sum -1$ diverges ($-1 \not\to 0$ BDT)

$\Rightarrow -4 < x < 4$ or $x \in (-4, 4)$. interval

19.

a. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n n!}{n^n}$

let $a_n = \frac{(-1)^n x^n n!}{n^n}$. By Ratio test, if this series converges, we have

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{n+1} \overbrace{(n+1)!}^{\to (n+1)(n!)}}{\underbrace{(n+1)^{n+1}}_{\to (n+1)(n+1)^n}} \cdot \frac{n^n}{(-1)^n x^n n!}\right|.$

$= \left|\frac{(-1)\cdot(n+1)(n!)}{(n+1)(n+1)^n} \cdot \frac{n^n}{n!} x\right| = \left|(-1)\left(\frac{n}{n+1}\right)^n x\right| \to \left|\frac{x}{e}\right| < 1.$

$\Rightarrow \left|\frac{x}{e}\right| < 1 \Rightarrow |x| < \underbrace{e}_{\text{radius}} \Rightarrow -e < x < e$

check $x = e$. $\sum_{n=1}^{\infty} \frac{(-1)^n e^n n!}{n^n}$ Diverges (BDT)

Since

$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \Rightarrow \underset{\downarrow\,\infty}{\sqrt{2\pi n}} \le \left(\frac{e^n}{n^n}\right) n! \le \underset{\downarrow\,\infty}{\sqrt{2\pi n}\cdot e^{\frac{1}{12n}}} \Rightarrow \lim_{n\to\infty} \frac{e^n n!}{n^n}$ Diverges

(Search Stirling formula)

check $x = -e$ $\sum_{n=1}^{\infty} \frac{(-1)^n (-e)^n n!}{n^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} e^n n!}{n^n} = \sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$ diverges.

(the same reason above)

$\Rightarrow -e < x < e$ or $\underline{\underline{x \in (-e, e)}}$ Interval

20. a. $y = (3x-1)^{\sin(x)}$

$$\Rightarrow \ln y = \sin(x) \ln(3x-1)$$

$$\xRightarrow{\frac{d}{dx}} \frac{y'}{y} = \cos(x)\ln(3x-1) + \sin(x) \cdot \frac{3}{3x-1}$$

$$\Rightarrow y' = \left[\cos(x)\ln(3x-1) + \sin(x)\frac{3}{3x-1}\right](3x-1)^{\sin(x)}$$

b. $y = (x+1)^{\ln x}$

$$\Rightarrow \ln y = \ln x \cdot \ln(x+1)$$

$$\xRightarrow{\frac{d}{dx}} \frac{y'}{y} = \frac{1}{x} \cdot \ln(x+1) + \ln x \cdot \frac{1}{x+1}$$

$$\Rightarrow y' = \left[\frac{\ln(x+1)}{x} + \frac{\ln x}{x+1}\right] \cdot (x+1)^{\ln x}.$$

c. $y = (x^2+2)^{\left(\frac{1}{\ln x}\right)}$

$$\Rightarrow \ln y = \left(\frac{1}{\ln x}\right) \cdot \ln(x^2+2)$$

$$\xRightarrow{\frac{d}{dx}} \frac{y'}{y} = \frac{-1}{x(\ln x)^2} \cdot \ln(x^2+2) + \frac{1}{\ln(x)} \cdot \frac{2x}{x^2+2}$$

$$\Rightarrow y' = \left[\frac{-\ln(x^2+2)}{x(\ln x)^2} + \frac{2x}{(x^2+2)\ln(x)}\right] \cdot (x^2+2)^{\left(\frac{1}{\ln x}\right)}$$

21. Determine the convergence or divergence for each series with the given general term:

Series	Converge or Diverge?	Test used
$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$	Diverge	$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}}$ P-series, $\frac{3}{4} \leq 1$
$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$	Diverge	B.D.T. $\frac{2^n}{n^3} \not\to 0$ as $n \to \infty$
$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n}\right)$	Converge	Telescoping, $b_n = \frac{-1}{n} \to 0$.
$\sum_{n=1}^{\infty} \frac{3^{2n}}{n!}$	Converge.	Ratio test. let $a_n = \frac{3^{2n}}{n!}$, $\frac{a_{n+1}}{a_n} = \frac{3^{2n+2}}{(n+1)!} \cdot \frac{n!}{3^{2n}} = \frac{3^2}{n+1} \to 0 < 1$ as $n \to \infty$
$\sum_{n=1}^{\infty} \cos(\pi n)$	Diverge.	BDT. $\cos(\pi n) \not\to 0$ as $n \to \infty$
$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$	Diverge	P-series $\sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, $\frac{1}{2} \leq 1$
$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{3n^3+1}$	Not absolutely converges but it converges conditionally	Alternating Series test. $\sum (-1)^{n-1} b_n$. $b_n = \frac{n^2}{3n^3+1} \downarrow 0$ and $b_n \to 0$.
$\sum_{n=0}^{\infty} 3\left(-\frac{1}{2}\right)^n$	Converges	Geometric series, $\lvert -\frac{1}{2} \rvert < 1$
V $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$	Converges	Integral test, $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \frac{-1}{\ln x}\Big\rvert_2^{\infty} = \frac{1}{\ln 2}$ Improper integral
$\sum_{n=1}^{\infty} ne^{-n^3}$	Converges	Root test, $a_n = \frac{n}{e^{n^3}}$, $\sqrt[n]{a_n} = \frac{\sqrt[n]{n}}{e^{n^2}} \to 0 < 1$
$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$	Diverges	Basic Divergence Test. $\left(\frac{n}{n+1}\right)^n \to \frac{1}{e} \neq 0$.
$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$	Converges	Limit Comparison with P-series $\sum \frac{1}{n^3}$
$\sum_{n=0}^{\infty} \left(\frac{2}{9}\right)^n$	Converges	Geometric series, $\lvert \frac{2}{9} \rvert < 1$.
$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$	Converges	Root test, $a_n = \frac{n^2}{2^n}$ $\sqrt[n]{a_n} = \frac{\sqrt[n]{n^2}}{2} \to \frac{1}{2} < 1$

$\sum_{n=1}^{\infty}(0.34)^n$	Converges	Geometric series $\|0.34\|<1$
$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$	Converges	P-series, $\frac{3}{2}>1$.
$\sum_{n=1}^{\infty}\frac{1}{2n+1}$	Diverges	Limit Comparison with P-series $\sum\frac{1}{n}$.

22. Find a parameterization of a line segment from $(4,4)$ to $(8,-5)$ and $t\in[0,1]$.