$$\lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^{2x} = \lim_{x\to\infty} 2x \ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{2x}} \left(\tfrac{0}{0}\right) \underset{L'}{=} \lim_{x\to\infty} \frac{\frac{x}{x+2}\cdot\left(\frac{-2}{x^2}\right)}{-\frac{1}{2}x^2} \to \lim_{x\to\infty} \frac{x}{x+2}\cdot 4 = 4$$

Math 1432

Exam 4 Review

Sol

1. In each of the following, determine whether or not L'Hopital's Rule applies. If it applies, state the indeterminate form then find the limit.

$\left(\frac{0}{0}\right)$ a. $\lim_{x\to 0} \frac{1+x-e^x}{x^2} \underset{L'}{\overset{(0/0)}{=}} \lim_{x\to 0} \frac{1-e^x}{2x} \underset{L'}{\overset{(0/0)}{=}} \lim_{x\to 0} \frac{-e^x}{2} = -\frac{1}{2}$

X b. $\lim_{x\to 1} \frac{x+\ln x}{2x^2} = \frac{1}{2}$

$(0\cdot\infty)$ c. $\lim_{x\to\pi/2} (x-\pi/2)\tan x = \lim_{x\to\frac{\pi}{2}} \frac{\left(x-\frac{\pi}{2}\right)\sin x}{\cos x} \underset{L'}{\overset{(0/0)}{=}} \lim_{x\to\frac{\pi}{2}} \frac{\sin x + \left(x-\frac{\pi}{2}\right)\cos x}{-\sin x} = -1$

(1^∞) d. $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{2x} = \lim_{x\to\infty} e^{\ln\left(1+\frac{2}{x}\right)^{2x}} = e^{\lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^{2x}} \overset{(*)}{=} e^4$

$\left(\frac{0}{0}\right)$ e. $\lim_{x\to 0} \frac{1-\cos x}{x^2} \underset{L'}{\overset{(0/0)}{=}} \lim_{x\to 0} \frac{+\sin x}{2x} \underset{L'}{\overset{(0/0)}{=}} \lim_{x\to 0} \frac{+\cos x}{2} = +\frac{1}{2}$

$(\infty-\infty)$ f. $\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right) = \lim_{x\to 0^+} \left(\frac{\sqrt{x}-x}{x\sqrt{x}}\right) = \lim_{x\to 0^+} \left(\frac{\sqrt{x}-x}{x^{\frac{3}{2}}}\right) \underset{L'}{\overset{(0/0)}{=}} \lim_{x\to 0^+} \frac{\left(\frac{1}{2}x^{-\frac{1}{2}}-1\right)x^{\frac{1}{2}}}{\left(\frac{3}{2}x^{\frac{1}{2}}\right)x^{\frac{1}{2}}} = \lim_{x\to 0^+} \frac{\frac{1}{2}-x^{\frac{1}{2}}}{\frac{3}{2}x^2}$ div

$\not\to 0$

$\left(\frac{\infty}{\infty}\right)$ g. $\lim_{n\to\infty} \frac{\ln(n+4)}{n+2} \underset{L'}{\overset{(\infty/\infty)}{=}} \lim_{n\to\infty} \frac{\frac{1}{n+4}}{1} = 0$

$$\boxed{\lim_{n\to\infty} \ln(3n)^{\frac{2}{n}} = \lim_{n\to\infty} \frac{2}{n}\ln(3n) \underset{L'}{\overset{(\infty/\infty)}{=}} \lim_{n\to\infty} \frac{\frac{2}{3n}}{1} = 0}$$

(∞^0) h. $\lim_{n\to\infty} (3n)^{\frac{2}{n}} = \lim_{n\to\infty} e^{\ln(3n)^{\frac{2}{n}}} = e^{\lim_{n\to\infty} \ln(3n)^{\frac{2}{n}}} = e^0 = 1$

(1^∞) i. $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{2n} = \lim_{n\to\infty} e^{\ln\left(1+\frac{3}{n}\right)^{2n}} = e^{\lim_{n\to\infty} \ln\left(1+\frac{3}{n}\right)^{2n}} = e^6$

$$\boxed{\lim_{n\to\infty} \ln\left(1+\frac{3}{n}\right)^{2n} = \lim_{n\to\infty} 2n \ln\left(1+\frac{3}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{2n}} = \lim_{n\to\infty} \frac{\frac{n}{3+n}\cdot\left(\frac{-3}{n^2}\right)}{\frac{-1}{2n^2}} = \lim_{n\to\infty} \frac{n}{3+n}\cdot 6 = 6}$$

$\left(\frac{\infty}{\infty}\right)$ j. $\lim_{x\to\infty} \frac{x^2}{\ln x} \underset{L'}{\overset{(\infty/\infty)}{=}} \lim_{x\to\infty} \frac{2x}{\frac{1}{x}} = \lim_{x\to\infty} 2x^2 \to \infty$ div.

(∞^0) k. $\lim_{x\to\infty} (e^{3x}+1)^{\frac{1}{2x}} = \lim_{x\to\infty} e^{\ln(e^{3x}+1)^{\frac{1}{2x}}} = e^{\lim_{x\to\infty} \ln(e^{3x}+1)^{\frac{1}{2x}}} = e^{\frac{3}{2}}$

$\left(\frac{0}{0}\right)$ l. $\lim_{x\to 0} \frac{\arctan(4x)}{x} \underset{L'}{\overset{(0/0)}{=}} \lim_{x\to 0} \frac{\frac{4}{1+(4x)^2}}{1} = 4$

$$\boxed{\lim_{x\to\infty} \ln(e^{3x}+1)^{\frac{1}{2x}} = \lim_{x\to\infty} \frac{1}{2x}\ln(e^{3x}+1) \underset{L'}{\overset{(\infty/\infty)}{=}} \lim_{x\to\infty} \frac{\frac{3e^{3x}}{e^{3x}+1}}{2} \underset{L'}{\overset{(\infty/\infty)}{=}} \lim_{x\to\infty} \frac{9e^{3x}}{2\cdot 3e^{3x}} = \frac{3}{2}}$$

2. Determine if each integral is improper. If it is improper, state why, re-write it using proper limit notation, and solve.

$x^{-\frac{2}{3}} \to \infty$ as $x\to 0$ a. $\int_0^{27} x^{-2/3} = \lim_{a\to 0} \int_a^{27} x^{-\frac{2}{3}}dx = \lim_{a\to 0} 3x^{\frac{1}{3}}\Big|_a^{27} = \lim_{a\to 0}\left[3\cdot 27^{\frac{1}{3}} - 3\cdot a^{\frac{1}{3}}\right] = 9$

$\frac{1}{\sqrt{4-x}} \to \infty$ as $x\to 4$ b. $\int_0^4 \frac{1}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^b \frac{1}{\sqrt{4-x}}dx = \lim_{b\to 4^-} -2\sqrt{4-x}\Big|_0^b = \lim_{b\to 4^-}\left[+2\sqrt{4} - 2\cancel{\sqrt{4-b}}\right] = +2\cdot 2 = +4$

$\frac{1}{(x-1)^{\frac{2}{3}}} \to \infty$ as $x\to 1$ c. $\int_1^9 (x-1)^{-2/3}dx = \lim_{a\to 1} \int_a^9 \frac{1}{(x-1)^{\frac{2}{3}}}dx = \lim_{a\to 1} 3(x-1)^{\frac{1}{3}}\Big|_a^9 = \lim_{a\to 1}\left[3\cdot 8^{\frac{1}{3}} - 3\cancel{(a-1)^{\frac{1}{3}}}\right] = 3\cdot 2 = 6$

$\frac{1}{\sqrt{x}} \to \infty$ as $x\to 0$ d. $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \lim_{a\to 0} \int_a^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \lim_{a\to 0} 2e^{\sqrt{x}}\Big|_a^4 = \lim_{a\to 0}\left[2e^{\sqrt{4}} - 2e^{\sqrt{a}}\right] = 2e^2 - 2$

$\boxed{e^x>0}$

X e. $\int_0^1 \frac{1}{e^x}\,dx = \int_0^1 e^{-x}dx = -e^{-x}\Big|_0^1 = -e^{-1}+e^0 = 1-\frac{1}{e}$

$\frac{1}{\sqrt{x-2}} \to \infty$ as $x\to 2$ f. $\int_2^6 \frac{1}{\sqrt{x-2}}\,dx = \lim_{a\to 2}\int_a^6 \frac{1}{\sqrt{x-2}}dx = \lim_{a\to2} 2\sqrt{x-2}\Big|_a^6 = \lim_{a\to2}\left[2\sqrt{6-2} - 2\cancel{\sqrt{a-2}}\right] = 2\cdot2=4$

Integral on an unbdd interval g. $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = 2\int_0^\infty \frac{1}{1+x^2}dx = 2\cdot\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}dx = 2\cdot\lim_{b\to\infty}\tan^{-1}x\Big|_0^b = 2\cdot\frac{\pi}{2}=\pi$

$\frac{1}{(x-1)^{\frac12}}>0$ as $x\in(2,5)$ X h. $\int_2^5 (x-1)^{-1/2}\,dx = 2(x-1)^{\frac12}\Big|_2^5 = \left[2(4)^{\frac12} - 2(1)^{\frac12}\right] = 4-2=2.$

3. The series $4-3+\frac{9}{4}-\frac{27}{16}+\dots$ is a geometric series. Find the general term, a_n, and write the sum in sigma notation. Does this series converge? If so, what is the sum?

$4-3+\frac94-\frac{27}{16}+\cdots$

$= 4 - 4\cdot\frac34 + (-1)^2 4\cdot\frac34\cdot\frac34 + (-1)^3 4\cdot\frac34\cdot\frac34\cdot\frac34 + \cdots$

$= \sum_{n=0}^\infty 4\cdot\left(-\frac34\right)^n$

$= \frac{4}{1-(-\frac34)} = \frac{4}{\frac74} = \frac{16}{7}$

(Geometric) 4. Find the sum of the following (if possible):

$r=-\frac34$, $|r|<1$; first term is $(-\frac34)^0=1$

a. $\sum_{k=0}^{\infty}\left(-\frac{3}{4}\right)^k = \frac{\text{first term}}{1-r} = \frac{1}{1-(-\frac34)} = \frac{1}{\frac74} = \frac47$

$r=\frac23$, first term $(\frac23)^0=1$

b. $\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{1-\frac23} = \frac{1}{\frac13} = 3$ (Geometric)

$r=\frac54>1$

c. $\sum_{k=0}^{\infty}\left(\frac{5}{4}\right)^{k-1}$ X diverges

d. $\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right) = \underbrace{\left(\frac12-\cancel{\frac14}\right)}_{n=2}+\underbrace{\left(\frac13-\cancel{\frac15}\right)}_{n=3}+\underbrace{\left(\cancel{\frac14}-\frac16\right)}_{n=4}+\underbrace{\left(\cancel{\frac15}-\frac17\right)}_{n=5}+\cdots = \frac12+\frac13=\frac56$

$r=\frac67<1$, first term $=\frac{6}{7^{-2}}=6\cdot7^2$

e. $\sum_{k=0}^{\infty}\frac{6^{k+1}}{7^{k-2}} = \sum\frac{6^{k-2}\cdot6^3}{7^{k-2}} = \sum 6^3\left(\frac67\right)^{k-2} = \frac{6\cdot7^2}{1-(\frac67)} = 6\cdot7^3$

5. Determine whether the given series converges or diverges; state which test you are using to determine convergence/divergence and show all work.

Converges a. $\sum\frac{k^2 2^k}{(k+1)!}$ By ratio Test. Let $a_k=\frac{k^2 2^k}{(k+1)!}$. $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^2 2^{k+1}}{(k+2)!}\cdot\frac{(k+1)!}{k^2 2^k} = \frac{2(k+1)^2}{(k+2)k^2}\to 0<1$ as $k\to\infty$

Diverges b. $\sum\frac{3^{k+1}}{(k+1)^2 e^k}$ By ratio Test. Let $a_k=\frac{3^{k+1}}{(k+1)^2e^k}$. $\frac{a_{k+1}}{a_k} = \frac{3^{k+2}}{(k+2)^2e^{k+1}}\cdot\frac{(k+1)^2e^k}{3^{k+1}} = \frac3e\left(\frac{k+1}{k+2}\right)^2\to\frac3e>1$ as $k\to\infty$ $(e\doteq2.71)$

Diverges c. $\sum\frac{\ln n}{n}$ By Integral Test, and $\int_1^\infty\frac{\ln x}{x}dx = \frac12(\ln x)^2\Big|_1^\infty \Rightarrow \infty$

Converges d. $\sum\frac{2n+1}{\sqrt{n^5+3n^3+1}} \approx \sum\frac{n}{\sqrt{n^5}} = \sum\frac{n}{n^{\frac52}} = \sum\frac{1}{n^{\frac32}}$ (by p-series $\frac32>1$) converges

By limit Comparison

Diverges e. $\sum\frac{4n^2+1}{n^3-n}\approx\sum\frac{n^2}{n^3}=\sum\frac1n$

Converges f. $\sum\frac{4n^2+1}{n^5-n}\approx\sum\frac{n^2}{n^5}=\sum\frac1{n^3}$ $(3>1)$

By Limit comparison

Diverges g. $\sum\left(1+\frac{1}{n}\right)^n$

$\Rightarrow$ $\left(1+\frac1n\right)^n\to e$ as $n\to\infty$ which is not 0.

By Basic Divergence Test.

Converges h. $\sum \frac{n^3}{3^n}$ By Root test let $a_n = \frac{n^3}{3^n}$, $\sqrt[n]{a_n} = \frac{(\sqrt[n]{n})^3}{3} \to \frac{1}{3} < 1$ as $n \to \infty$

Diverges i. $\sum \frac{1}{\sqrt[4]{n^3}} = \sum \frac{1}{n^{\frac{3}{4}}}$ diverges by p-series Test. $(\frac{3}{4} < 1)$

6. Determine if the following series (A) converge absolutely, (B) converge conditionally or (C) diverge.

(B) a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$ (A) NOT abs. Consider $\sum \frac{\sqrt{n}}{n+3} \approx \sum \frac{\sqrt{n}}{n} = \sum \frac{1}{n^{\frac{1}{2}}}$ div. (B) It's Cond. by $\frac{\sqrt{n}}{n+3} \to 0$ as $n \to \infty$ Alternating Series Test

$\cos \pi n = (-1)^n$

(A) b. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2} = \sum \frac{(-1)^n}{n^2}$ (A) Consider $\sum \frac{1}{n^2} \Rightarrow$ converges $\Rightarrow$ It's abs. convergent.

(B) c. $\sum_{n=0}^{\infty} \frac{4n(-1)^n}{3n^2+2n+1}$ (A) NOT abs. Consider $\sum \frac{4n}{3n^2+2n+1} \approx \sum \frac{n}{n^2} = \sum \frac{1}{n}$ div (B) $\frac{4n}{3n^2+2n+1} \to 0$ as $n \to \infty$ It's cond. by Alternating Series Test

(B) d. $\sum_{n=0}^{\infty} \frac{3(-1)^n}{\sqrt{3n^2+2n+1}}$ (A) NOT abs. Consider $\sum \frac{3}{\sqrt{3n^2+2n+1}} \approx \sum \frac{1}{\sqrt{n^2}} = \sum \frac{1}{n}$ div (B) $\frac{3}{\sqrt{3n^2+2n+1}} \to 0$ as $n \to \infty$ It's cond. by AST

(C) e. $\sum_{n=0}^{\infty} \frac{3n(-1)^n}{\sqrt{3n^2+2n+1}}$ (A) NOT abs. consider $\sum \frac{3n}{\sqrt{3n^2+2n+1}} \approx \sum \frac{n}{\sqrt{n^2}} = \sum 1$ div. (B) $\frac{3n}{\sqrt{3n^2+2n+1}} \to \sqrt{3} \neq 0$ DIV.

$e^x = \sum_{x=0}^{\infty} \frac{x^n}{n!}$

$e^{-x} = \sum_{x=0}^{\infty} \frac{(-x)^n}{n!}$

7. Use the Taylor series expansion (in powers of x) for $f(x) = e^x$ to find the Taylor series expansion $f(x) = \cosh x$. $= \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\sum_{x=0}^{\infty} \frac{x^n}{n!} + \frac{(-x)^n}{n!}\right) = \frac{1}{2}\left(\sum_{x=0}^{\infty} \frac{x^n + (-x)^n}{n!}\right)$

$\cos x \doteq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

8. Determine the Taylor polynomial in powers of x of degree 8 for the function $f(x) = x - \cos(x^2)$. $= x - \left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!}\right) = -1 + x + \frac{x^4}{2!} - \frac{x^8}{4!}$

9. Determine the Taylor polynomial in powers of x of degree 5 for the function $f(x) = \frac{1 - e^x}{x}$ $\Rightarrow \frac{1 - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}\right)}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \frac{x^5}{6!}$

$f'(x) = 2\cos 2x$, $f'(\pi) = 2$

$f''(x) = -4\sin 2x$, $f''(\pi) = 0$

$f'''(x) = -8\cos 2x$, $f'''(\pi) = -8$

$f^{(4)}(x) = 32\sin 2x$, $f^{(4)}(\pi) = 0$

10. Determine the Taylor polynomial in powers of $x - \pi$ of degree 4 for the function $f(x) = \sin(2x)$. $f(\pi) = 0$ $\Rightarrow$ $P_4 = \frac{2}{1!}(x - \pi) - \frac{8}{3!}(x - \pi)^3$

11. Assume that f is a function such that $|f^{(n)}(x)| \leq 2$ for all n and x.

a. Estimate the maximum possible error if $P_4(0.5)$ is used to approximate $f(0.5)$

b. Find the least integer n for which $P_n(0.5)$ approximates $f(0.5)$ with an error less than 10^{-3}.

12. Use the values in the table below and the formula for Taylor polynomials to give the 5th degree Taylor polynomial for f centered at $x = 0$.

$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$
1	0	-2	3	-4	1

11. (a) $\circledast$ $|f^{(n)}(x)| \le 2 \quad \forall x, \forall n$ is given.

$$R_4(0.5) = \left| \frac{f^{(5)}(c)}{5!}(0.5)^5 \right| , \quad c \in (0, 0.5) = (0, \tfrac{1}{2})$$

$$= |f^{(5)}(c)| \left| \frac{1}{5!}\left(\frac{1}{2}\right)^5 \right| \overset{\circledast}{\le} 2 \left| \frac{1}{5!}\left(\frac{1}{2}\right)^5 \right| = \frac{1}{5!} \cdot \frac{1}{2^4}$$

(b) Find n s.t. $|R_n(0.5)| \le 10^{-3} = \frac{1}{1000}$

$$\Rightarrow |R_n(0.5)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} \right| = |f^{(n+1)}(c)| \cdot \left| \frac{1}{(n+1)!} \cdot \frac{1}{2^{n+1}} \right|.$$

$$\overset{\circledast}{\le} 2 \cdot \frac{1}{(n+1)!} \cdot \frac{1}{2^{n+1}} = \boxed{\frac{1}{(n+1)!} \cdot \frac{1}{2^n} \le \frac{1}{1000}}$$

$1000 \le 2^n \cdot (n+1)!$ Try $n=3$, $2^3(3+1)! = 8 \cdot 24 < 1000$ ✗

$\boxed{n=4}$ $2^4(4+1)! = 16 \cdot 120 > 1000$ ✓

12. $$P_5 = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$= 1 + 0 \cdot x + \frac{-2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{-4}{4!}x^4 + \frac{1}{5!}x^5$$

$$= 1 - x^2 + \frac{1}{2}x^3 - \frac{1}{3!}x^4 + \frac{1}{5!}x^5$$