Math 1432 Exam 2 Review

1. (a) ① $f(x)=x^3+1$, $\Rightarrow f'(x)=3x^2>0 \Rightarrow$ 1-1 ✓
   ② Find f^{-1}. $\Rightarrow x=y^3+1 \Rightarrow y^3=x-1. \Rightarrow \boxed{y=\sqrt[3]{x-1}}$

   (b) ① $f(x)=3x+10$, $f'(x)=3>0 \quad \forall x\in\mathbb{R} \Rightarrow$ 1-1 ✓
   ② Find f^{-1}. $x=3y+10 \Rightarrow \boxed{y=\frac{1}{3}(x-10)}$

   (c) ① $f(x)=\sqrt{9-x^2}$ $(-3<x<3)$ $\Rightarrow f'(x)=\frac{-x}{\sqrt{9-x^2}} \Rightarrow$ not 1-1

2. By Thm. if f has an inverse, and $f(a)=b$, $f'(a)\neq 0$
   Then $[f^{-1}(b)]'=\frac{1}{f'(a)}$
   Now, $\underset{a}{f(3)}=\underset{b}{1}$, $\underset{a}{f'(3)}=\frac{2}{7}$, then $(f^{-1})'(1)=\frac{1}{f'(3)}=\boxed{\frac{7}{2}}$

3. By Thm above, Since f passes through $(-1,2)$, that is, $\boxed{f(-1)=2}$
   and the slope of ~~tangent line to the graph of f~~ at $x=-1$
   is $7/2$, that is $\boxed{f'(-1)=\frac{7}{2}}$
   Then $(f^{-1})'(2)=\frac{1}{f'(-1)}=\boxed{\frac{2}{7}}$

4. First, Find x s.t. $f(x)=9$. we have $x^3+1=9 \Rightarrow x^3=8$, $x=2$.
   $\Rightarrow \boxed{f(2)=9}$, Then we have $\boxed{f'(2)}=3(2)^2=\boxed{12}$
   Thus, $(f^{-1})'(9)=\frac{1}{f'(2)}=\boxed{\frac{1}{12}}$

5. Find the derivative.

a. $y = \ln\sqrt{e^x+4x} \Rightarrow y' = \dfrac{e^x+4}{2(e^x+4x)}$

$\left(\text{or } y = \ln(e^x+4x)^{\frac{1}{2}} = \frac{1}{2}\ln(e^x+4x)\right)$

b. $y = \sin(\ln(5-x)^6) = \sin(6\ln(5-x))$

$\Rightarrow y' = \dfrac{-6}{5-x}\cos(6\ln(5-x))$

c. $y = x^2e^{2x} + \cancel{\ln e^{2x}}\,2x \Rightarrow y' = 2xe^{2x} + 2x^2e^{2x} + 2$.

d. $y = e^{x^2}\cosh(3x), \Rightarrow y' = 2xe^{x^2}\cosh(3x) + 3e^{x^2}\sinh(3x)$

e. $f(x) = \ln(\sec\sqrt{x}) \Rightarrow f'(x) = \dfrac{1}{\sec\sqrt{x}} \cdot (\sec\sqrt{x})'$

$= \dfrac{1}{\sec\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}}\sec\sqrt{x}\cdot\tan\sqrt{x} = \dfrac{\tan\sqrt{x}}{2\sqrt{x}}$

f. $f(x) = \dfrac{e^{\sqrt{x}}}{x^3} \Rightarrow f'(x) = \dfrac{\frac{x^3}{2\sqrt{x}}e^{\sqrt{x}} - 3x^2e^{\sqrt{x}}}{x^6}$

g. $y = (\cos x)^{(x+1)} = e^{\ln(\cos x)^{(x+1)}} = e^{(x+1)\ln(\cos x)}$.

$y' = [(x+1)\ln(\cos x)]' \cdot e^{(x+1)\ln(\cos x)}$

$= \left[\ln(\cos x) + \dfrac{x+1}{\cos x}\cdot(-\sin x)\right]\cdot(\cos x)^{(x+1)}$

h. $f(x) = (3x-1)^{(2x+6)} = e^{\ln(3x-1)^{(2x+6)}} = e^{(2x+6)\ln(3x-1)}$

$f'(x) = \left[2\ln(3x-1) + \dfrac{2x+6}{3x-1}\cdot 3\right](3x-1)^{(2x+6)}$

③

5. (i). $f(x) = \ln(5x^2) + e^{6x} + \arctan(5-2x)$.

$$f'(x) = \frac{10x}{5x^2} + 6e^{6x} + \frac{-2}{1+(5-2x)^2}$$

(j). $f(x) = \log_7(3x^2) = \dfrac{\ln(3x)^2}{\ln 7} = \dfrac{1}{\ln 7} \cdot 2\ln(3x)$.

$$f'(x) = \frac{2}{\ln 7} \cdot \frac{1}{x}$$

(k). $y = 6^{-2x} = e^{\ln 6^{-2x}} = e^{-2x\ln 6}$. , $y' = -2(\ln 6)\, 6^{-2x}$

(l). $f(x) = \arctan(2x^3) \Rightarrow f'(x) = \dfrac{1}{1+(2x^3)^2} \cdot 6x^2$

6. Integrate:

(a) $\displaystyle\int_e^{4e} \frac{1}{x}\,dx = \ln|x| \Big|_e^{4e} = \ln 4e - \ln e = 4 - 1 = 3$.

(b) $\displaystyle\int \left(\frac{\csc^2 x}{2+5\cot x} - e^{9x}\right)dx = \int \frac{\csc^2 x}{2+5\cot x}\,dx - \int e^{9x}\,dx$

let $u = 2+5\cot x$
$du = -5\csc^2 x\,dx$

$$= -\frac{1}{5}\int \frac{du}{u} - \frac{1}{9}e^{9x} + C.$$

$$= -\frac{1}{5}\ln|u| - \frac{1}{9}e^{9x} + C.$$

$$= -\frac{1}{5}\ln|2+5\cot x| - \frac{1}{9}e^{9x} + C.$$

let $u = 2+\cosh x$
$du = \sinh x\, dx$

6. (c) $\displaystyle\int \frac{\sinh x}{(2+\cosh x)^2}\,dx \overset{\downarrow}{=} \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{2+\cosh x} + C.$

(d) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C.$

(e) $\displaystyle\int \frac{2}{\sqrt{x}(3-\sqrt{x})}\,dx = -4\int \frac{du}{u} = -4\ln|u| + C$

$\displaystyle = -4\ln|3-\sqrt{x}| + C.$

let $u = 3-\sqrt{x}$
$\displaystyle du = -\frac{1}{2\sqrt{x}}\,dx$

(f) $\displaystyle\int \frac{\overset{=(x+1)+1}{x+2}}{x+1}\,dx = \int \frac{1}{x+1} + \frac{x+1}{x+1}\,dx = \int \frac{1}{x+1} + 1\,dx$

$= \ln|x+1| + x + C.$

(g) $\displaystyle\int \frac{3x^2+3x+3}{x^2+1}\,dx = \int \frac{3x^2+3}{x^2+1} + \frac{3x}{x^2+1}\,dx = \int 3 + \frac{3x}{x^2+1}\,dx.$

$\displaystyle = 3x + \frac{3}{2}\ln|x^2+1| + C$

(h) $\displaystyle\int \frac{\cos^3 x - \sin^2 x}{\cos^2 x}\,dx = \int \cos x - \underbrace{\tan^2 x}_{} \overset{(\sec^2-1)}{\,} dx$

$\displaystyle = \int \cos x - \sec^2 x + 1\,dx = \sin x - \tan x + x + C.$

(i) $\displaystyle\int \tan(3x)\,dx = \frac{1}{3}\ln|\sec(3x)| + C.$

(j) $\displaystyle\int \frac{\arctan(3x)}{1+9x^2}\,dx = \frac{1}{3}\int u\,du = \frac{1}{3}\frac{u^2}{2} + C = \frac{1}{3}\frac{(\tan(3x))^2}{2} + C$

$\displaystyle u = \arctan(3x) \Rightarrow du = \frac{3\,dx}{1+9x^2}$ $\displaystyle = \frac{(\tan(3x))^2}{6} + C$

6. (k) $\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x \Big|_0^{\frac{\sqrt{3}}{2}} = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

(l). $\int \cos^4x \sin^3x\,dx = \int \cos^4x\,(1-\cos^2x)\sin x\,dx$. (let $u=\cos x$, $du=-\sin x\,dx$)

$= -\int u^4(1-u^2)\,du = -\frac{u^5}{5} + \frac{u^7}{7} + C$

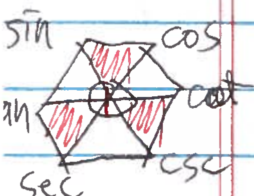
$= -\frac{(\cos x)^5}{5} + \frac{(\cos x)^7}{7} + C$

(m) $\int \cos^5x \sin^2x\,dx = \int \cos x\,(1-\sin^2x)^2\sin^2x\,dx$ (let $u=\sin x$, $du=\cos x\,dx$)

$= \int (1-u^2)^2u^2\,du = \int u^2(u^4-2u^2+1)\,du$.

$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C = \frac{(\sin x)^7}{7} - \frac{2}{5}(\sin x)^5 + \frac{(\sin x)^3}{3} + C$

(n) $\int \cot^3x\,dx = \int \cot x\cdot(\csc^2x - 1)\,dx = \int(\cot x\cdot\csc^2x - \cot x)\,dx$



$= -\frac{1}{2}(\cot x)^2 - \ln|\sin x| + C$.

(o) $\int \underbrace{x}_{A}\,\underbrace{\ln(2x)}_{L}\,dx = \frac{x^2}{2}\ln(2x) - \int \frac{x}{2}\,dx$

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$u=\ln(2x)$ $dv = x\,dx$

$du = \frac{dx}{x}$ $v = \frac{x^2}{2}$

$= \frac{x^2}{2}\ln(2x) - \frac{x^2}{4} + C$.

6. (p) $\int \underset{A}{2x} \underset{T}{\sin(3x)}\,dx = -\frac{2x}{3}\cos(3x) + \frac{2}{9}\sin(3x) + C$

u	dv	
$2x$	$\sin(3x)$	$+$
2	$-\frac{1}{3}\cos(3x)$	$-$
0	$-\frac{1}{9}\sin(3x)$	$+$
		$-$

(q) $\int \frac{5}{\underset{\underset{6^2}{=}}{36}+(x-1)^2}\,dx = 5\cdot\frac{1}{6}\tan^{-1}\left(\frac{x-1}{6}\right) + C$

(r) $\int \tan^4(x)\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx$

$= \int \tan^2 x\sec^2 x\,dx - \int \underbrace{\tan^2 x}_{= \sec^2 x - 1}\,dx$

$= \int \tan^2 x\sec^2 x\,dx - \int \sec^2 x\,dx + \int dx = \frac{(\tan x)^3}{3} - \tan x + x + C$

(s) $\int 2x\sec(4x^2)\,dx = \frac{1}{4}\int \sec u\,du = \frac{1}{4}\ln|\sec u + \tan u| + C$

let $u = 4x^2$,
$du = 8x\,dx$

$= \frac{1}{4}\ln|\sec(4x^2) + \tan(4x^2)| + C$

(t). $\int \sec^4 x\,dx = \int \sec^2 x(1+\tan^2 x)\,dx = \int \sec^2 x\,dx + \int \tan^2 x\sec^2 x\,dx$

$= \tan x + \frac{(\tan x)^3}{3} + C$

7. $\frac{dy}{dx} = (y+5)(x+2) \Rightarrow \frac{dy}{y+5} = (x+2)dx$

Take "$\int$" $\Rightarrow$ $\int \frac{dy}{y+5} = \int (x+2)dx \Rightarrow \ln|y+5| = \frac{x^2}{2} + 2x + C'$

(or "take e" $y+5 = C_0 e^{\frac{x^2}{2}+2x}$).

8. $\begin{cases} \frac{dy}{dx} = y-2 \\ y(0) = 6 \end{cases} \Rightarrow \frac{dy}{y-2} = dx \xRightarrow{\text{"}\int\text{"}} \ln|y-2| = x + C'$

$\Rightarrow y-2 = Ce^x$

since $y(0)=6 \Rightarrow \cancel{\ln|} \; 6-2 = Ce^0 = C \Rightarrow C=4.$

$\Rightarrow y = 4e^x + 2.$

9. Given: $N_t = 200e^{kt}$, $N=300$ as $t=4$.

① $k=?$ $300 = 200 \cdot e^{4k} \Rightarrow \frac{3}{2} = e^{4k} \Rightarrow k = \frac{1}{4} \ln(\frac{3}{2}).$

② Triple in size $\Rightarrow 3 = e^{kt} \Rightarrow 3 = e^{\frac{1}{4}\ln(\frac{3}{2})t}$

$\Rightarrow \ln 3 = \frac{1}{4}\ln(\frac{3}{2})t \Rightarrow t = \frac{4\ln(3)}{\ln(\frac{3}{2})}$

10. Initial: $P_0 \Rightarrow 2P_0$ after 12500 years.

$P(t) = P_0 e^{kt} \Rightarrow 2P_0 = P(12500) = P_0 \cdot e^{k \cdot 12500}$

$\Rightarrow 2 = e^{k \cdot 12500}$, $k = \frac{\ln(2)}{12500}$

93% more than now $\Rightarrow (1+0.93)P_0 = P_0 \cdot e^{kt} \Rightarrow 1.93 = e^{kt}$

$t = \frac{\ln(1.93)}{k} = \frac{12500 \ln(1.93)}{\ln 2}$

11. Initial: $P_0 \Rightarrow P(t)=P_0 e^{kt}$
Double in 15 years $\Rightarrow P(15)=2P_0$. $\Big] \Rightarrow 2P_0 = P(15) = P_0 e^{k \cdot 15}$

$\Rightarrow \ln(2) = k \cdot 15 \Rightarrow k = \frac{\ln 2}{15}$.

12. $f(x) = \ln(2x-5) + e^{x-3}$

$f'(x) = \frac{2}{2x-5} + e^{x-3}$.

@ $(3,1)$, $\Rightarrow$ Slope of tangent line $\overset{m_t}{=}$ is $f'(3) = 2+1 = 3$.

Slope of normal line is $-\frac{1}{f'(3)} = -\frac{1}{3}$
$\overset{\shortparallel}{m_n}$ $(m_t \cdot m_n = -1)$

Equation of tangent: $(y-1) = 3(x-3)$

" " normal: $(y-1) = -\frac{1}{3}(x-3)$