# Math 1432, Section 12869
## Spring 2014
### Homework Assignment 14
### Due Date: 4/28/14 in Lab

Name: Sol

ID: ____________

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Instructions

- Print out this file and complete the problems. You must do all the problems!
- If the problem is from the text, the section number and problem number are in parantheses
- Use a blue or black pen or a pencil (dark).
- Write your solutions in the spaces provided. You must show work in order receive credit for a problem.
- Remember that your homework must be complete, neatly written and stapled.
- Submit the completed assignment to your Teaching Assistant in lab on the due date.
- If you do not do all of the problems, then your recitation quiz from the previous Friday will automatically become a ZERO.

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**1.** (Section 11.7, Problem 3)

- $\sum k x^k$

let $a_k = kx^k$. By Root Test,

$\sqrt[k]{|a_k|} = \sqrt[k]{k}\cdot|x| \to |x|$ as $k\to\infty$

So $\sum kx^k$ converges $\Leftrightarrow \sqrt[k]{|a_k|} \to |x| < 1$

$\Rightarrow \boxed{-1 < x < 1}$

- Check endpoints: $x=1$. $\Rightarrow \sum k\cdot 1^k = \sum k$ diverges

$x=-1 \Rightarrow \sum k(-1)^k$ diverges since $k(-1)^k \not\to 0$ as $k\to\infty$

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**2.** (Section 11.7, Problem 5)

- $\sum \frac{1}{(2k)!}x^k$ converges $\Leftrightarrow$ By Ratio test Let $a_k = \frac{x^k}{(2k)!}$.

$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right| = \left|\frac{x}{k+1}\right| \to 0 < 1$ as $k\to\infty$

$\Rightarrow \boxed{x\in(-\infty,\infty)}$

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**3.** (Section 11.7, Problem 6)

- $\sum \frac{2^k}{k^2}x^k$ converges $\Leftrightarrow$ By Root Test Let $a_k = \frac{2^k}{k^2}x^k$

$\sqrt[k]{|a_k|} = \frac{\sqrt[k]{|2^k x^k|}}{\sqrt[k]{k^2}} = \frac{|2x|}{\underbrace{(\sqrt[k]{k})^2}_{\to 1 \text{ as } k\to\infty}} \to |2x| < 1$ as $k\to\infty$

and $|x| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

- Check endpoints: as $x=\frac{1}{2}$, $\sum \frac{2^k}{k^2}\cdot\left(\frac{1}{2}\right)^k = \sum\frac{1}{k^2}$ converges

as $x=-\frac{1}{2}$ $\sum\frac{2^k}{k^2}\cdot\left(-\frac{1}{2}\right)^k = \sum\frac{(-1)^k}{k^2}$ converges

$\Rightarrow \boxed{-\frac{1}{2}\le x\le\frac{1}{2}}$ or $\boxed{x\in\left[-\frac{1}{2},\frac{1}{2}\right]}$

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**4.** (Section 11.7, Problem 9)

- $\sum \frac{1}{k2^k}x^k$ converges

By Root Test $\Leftrightarrow$ Let $a_k = \frac{x^k}{k2^k}$, $\sqrt[k]{|a_k|} = \frac{\sqrt[k]{|x|^k}}{\sqrt[k]{k\cdot 2^k}} = \frac{|x|}{\underbrace{\sqrt[k]{k}}_{\to 1 \text{ as } k\to\infty}\cdot 2} \to \frac{|x|}{2}$ as $k\to\infty$

and $\frac{|x|}{2} < 1 \Rightarrow |x| < 2 \Rightarrow -2 < x < 2$

- Check endpoints: as $x=2$, $\sum\frac{1}{k2^k}\cdot 2^k = \sum\frac{1}{k}$ diverges

as $x=-2$, $\sum\frac{1}{k(2^k)}\cdot(-2)^k = \sum\frac{(-1)^k}{k}$ converges (by Alternating Series Test)

$\Rightarrow \boxed{-2\le x<2}$ or $\boxed{x\in[-2,2)}$

(*) $\sqrt[k]{3} = 3^{\frac{1}{k}} \to 3^0 = 1$ as $k\to\infty$

(✩) $(-1)^k(-1)^k = (-1)^{2k} = [(-1)^2]^k = 1^k = 1$

5. (Section 11.7, Problem 10) By Root Test

• $\sum \frac{1}{k^2 2^k} x^k$ converges $\Leftrightarrow$ Let $a_k = \frac{x^k}{k^2 2^k}$, $\sqrt[k]{|a_k|} = \frac{\sqrt[k]{|x|^k}}{\sqrt[k]{k^2 2^k}}$

$= \frac{|x|}{(\sqrt[k]{k})^2 \cdot 2} \to \frac{|x|}{2}$ as $k\to\infty$ and $\frac{|x|}{2} < 1 \Rightarrow |x| < 2$. $(-2 < x < 2)$

• Check the endpoints: as $x=2$, we have $\sum \frac{2^k}{k^2 2^k} = \sum \frac{1}{k^2}$ converges

and as $x=-2$, $\sum \frac{1}{k^2 2^k}(-2)^k = \sum \frac{(-1)^k}{k^2}$ converges (by Alternating Series Test)

$\Rightarrow$ $\boxed{-2 \le x \le 2}$ or $\boxed{x \in [-2,2]}$

6. (Section 11.7, Problem 15) By Root Test

∘ $\sum \frac{k-1}{k} x^k$ converges $\Leftrightarrow$ Let $a_k = \frac{k-1}{k} x^k$, $\sqrt[k]{|a_k|} = \sqrt[k]{\frac{k-1}{k}|x|^k}$

$= \frac{\sqrt[k]{k-1}}{\sqrt[k]{k}} \cdot |x| \to |x|$ as $k\to\infty$ and $|x| < 1 \Rightarrow -1 < x < 1$

∘ Check endpoints: $x=1$, $\sum \frac{k-1}{k}$ diverges since $\frac{k-1}{k} \to 1 \neq 0$ as $k\to\infty$ (Basic Divergence Test)

$x=-1$, $\sum \frac{k-1}{k}(-1)^k$ diverges since $\frac{k-1}{k}(-1)^k \not\to 0$ as $k\to\infty$.

$\Rightarrow x \in (-1,1)$

7. (Section 11.7, Problem 18) By Root Test

• $\sum \frac{3k^2}{e^k} x^k$ converges $\Leftrightarrow$ Let $a_k = \frac{3k^2}{e^k} x^k$, $\sqrt[k]{|a_k|} = \sqrt[k]{\frac{3k^2}{e^k}} \cdot |x|$

$= \frac{\sqrt[k]{3} \cdot (\sqrt[k]{k})^2}{e} |x| \xrightarrow{(*)} \frac{|x|}{e}$ and $\frac{|x|}{e} < 1 \Rightarrow |x| < e$ or $-e < x < e$

∘ Check the endpts, as $x=e$, $\sum \frac{3k^2}{e^k} e^k = \sum 3k^2$ diverges $(3k^2 \to \infty \neq 0$ as $k\to\infty)$

as $x=-e$, $\sum \frac{3k^2}{e^k}(-e)^k = \sum (-1)^k 3k^2$ diverges

$\Rightarrow \boxed{-e < x < e}$ or $\boxed{x \in (-e,e)}$

8. (Section 11.7, Problem 21) By Root Test

∘ $\sum \frac{(-1)^k}{k^k}(x-2)^k$ converges $\Leftrightarrow$ Let $a_k = \frac{(-1)^k}{k^k}(x-2)^k$,

$\sqrt[k]{|a_k|} = \sqrt[k]{\left|\frac{(-1)^k}{k^k}\right| |x-2|^k} = \frac{1}{\sqrt[k]{k^k}} \cdot |x-2| \to |x-2|$ as $k\to\infty$

and $|x-2| < 1 \Rightarrow 1 < x < 3$

∘ Check endpts. as $x=3$, $\sum \frac{(-1)^k}{k^k}(3-2)^k = \sum \frac{(-1)^k 1^k}{k^k} = \sum \frac{(-1)^k}{k^k}$ div. $\left(\frac{(-1)^k}{k^k} \not\to 0 \text{ as } k\to\infty\right)$

as $x=1$, $\sum \frac{(-1)^k(1-2)^k}{k^k} = \sum \frac{(-1)^k(-1)^k}{k^k} \overset{(✩)}{=} \sum \frac{1}{k^k}$ div. $\left(\frac{1}{k^k} \not\to 0 \text{ as } k\to\infty\right)$

$\Rightarrow \boxed{1 < x < 3}$ or $\boxed{x \in (1,3)}$

9. (Section 11.7, Problem 22) By Ratio Test

$\sum k! x^k$ converges $\Leftrightarrow$ Let $a_k = k! x^k$.

$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)!}{k!} \frac{x^{k+1}}{x^k}\right| = |k+1||x| \to \infty > 1$ $\forall x$,

Thus, this series converges only at $x=0$

10. (Section 11.7, Problem 23) By Ratio Test

∘ $\sum (-1)^k \frac{2^k}{3^{k+1}} x^k$ converges $\Leftrightarrow$ Let $a_k = \frac{(-1)^k}{3^{k+1}} 2^k x^k$

$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+1} 2^{k+1}}{3^{k+2}} x^{k+1} \cdot \frac{3^{k+1}}{(-1)^k 2^k} \frac{1}{x^k}\right| = \left|\frac{-1}{3} \cdot 2x\right| = \left|\frac{2x}{3}\right|$

and $\left|\frac{2x}{3}\right| < 1 \Rightarrow |x| < \frac{3}{2} \Rightarrow -\frac{3}{2} < x < \frac{3}{2}$

∘ check endpts. As $x = \frac{3}{2}$, $\sum (-1)^k \frac{2^k}{3^{k+1}}\left(\frac{3}{2}\right)^k = \sum (-1)^k \frac{1}{3}$ diverges $\left((-1)^k \frac{1}{3} \not\to 0 \text{ as } k\to\infty\right)$

and as $x = -\frac{3}{2}$, $\sum (-1)^k \frac{2^k}{3^{k+1}}\left(-\frac{3}{2}\right)^k = \sum (-1)^{2k} \frac{1}{3} = \sum \frac{1}{3}$ diverges

$\Rightarrow \boxed{-\frac{3}{2} < x < \frac{3}{2}}$ or $\boxed{x \in \left(-\frac{3}{2}, \frac{3}{2}\right)}$

By Ratio Test

$\sum (-1)^k \frac{k!}{k^3}(x-1)^k$ converges $\Leftrightarrow$ Let $a_k = (-1)^k \frac{k!}{k^3}(x-1)^k$

and $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+1}(k+1)!}{(k+1)^3}(x-1)^{k+1} \cdot \frac{k^3}{(-1)^k k!} \cdot \frac{1}{(x-1)^k}\right|$

11. (Section 11.7, Problem 25)

$= \left|\frac{(-1)(k+1)}{(k+1)^3} \cdot k^3\right| |x-1| = \left|\frac{(-1)k^3}{(k+1)^2}\right| |x-1| \to \infty$ as $k \to \infty$

$\Rightarrow \infty$ as $k \to \infty$

So this series converges only as $x-1=0$ i.e. $x=1$

By Ratio Test

$\sum (-1)^k \frac{k^2}{(k+1)!}(x+3)^k$. converges $\Leftrightarrow$ Let $a_k = \frac{(-1)^k k^2}{(k+1)!}(x+3)^k$

12. (Section 11.7, Problem 29)

and $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+1}(k+1)^2}{(k+2)!}(x+3)^{k+1} \cdot \frac{(k+1)!}{(-1)^k k^2} \cdot \frac{1}{(x+3)^k}\right|$

$= \left|\frac{(-1)}{k+2} \cdot \left(\frac{k+1}{k}\right)^2\right| |x+3| \to 0$ as $k \to \infty$

which is always less than 1. for all $x$ $\Rightarrow 0$ as $k \to \infty$ (deg of Top < deg of bottom)

So $x \in (-\infty, \infty)$.

13. (Section 11.7, Problem 30)

By Root Test

$\circ \sum \frac{k^3}{e^k}(x-4)^k$ converges $\Leftrightarrow$ Let $a_k = \frac{k^3}{e^k}(x-4)^k$.

$\sqrt[k]{|a_k|} = \sqrt[k]{\frac{k^3}{e^k}|x-4|^k} = \frac{(\sqrt[k]{k})^3}{e}|x-4| \to \frac{|x-4|}{e}$ as $k \to \infty$

and $\frac{|x-4|}{e} < 1 \Rightarrow |x-4| < e \Rightarrow 4-e < x < e+4$.

$\circ$ Check endpoint. As $x = e+4 \Rightarrow \sum \frac{k^3}{e^k}(e+4-4)^k = \sum k^3 \frac{e^k}{e^k} = \sum k^3$

As $x = 4-e \Rightarrow \sum \frac{k^3}{e^k}(4-e-4)^k = \sum (-1)^k k^3$ diverges diverges

$\Rightarrow \boxed{4-e < x < e+4}$

$\circ$ $1 - \frac{x}{2} + \frac{2x^2}{4} - \frac{3x^3}{8} + \frac{4x^4}{16} + \cdots$ pattern: $(-1)^k \frac{k}{2^k} x^k$

$2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4$ $\qquad k = ,1,2,\cdots$

14. (Section 11.7, Problem 37)

By Root Test

$= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{k}{2^k} x^k$ converges $\Leftrightarrow$ Let $a_k = (-1)^k \frac{k}{2^k} x^k$

$\sqrt[k]{|a_k|} = \sqrt[k]{\left|(-1)^k \cdot \frac{k}{2^k}\right| \cdot |x|^k} = \frac{\sqrt[k]{k}}{2} \cdot |x| \to \frac{|x|}{2}$ as $k \to \infty$

and $\frac{|x|}{2} < 1 \Rightarrow -2 < x < 2$

$\circ$ check endpt. As $x = 2$. $1 + \sum (-1)^k \frac{k}{2^k} \cdot 2^k = 1 + \sum (-1)^k k$ diverges

As $x = -2$, $1 + \sum (-1)^k \frac{k}{2^k}(-2)^k = 1 + \sum k$ div.

$\Rightarrow \boxed{-2 < x < 2}$

15. (Section 11.7, Problem 39)

$\circ$ $\frac{3x^2}{4} + \frac{9x^4}{9} + \frac{27x^6}{16} + \frac{81x^8}{25} + \cdots = \sum_{k=1}^{\infty} \frac{3^k}{(k+1)^2} x^{2k}$ converges

By Root test

$\Leftrightarrow$ Let $a_k = \frac{3^k}{(k+1)^2} x^{2k}$ $\sqrt[k]{|a_k|} = \sqrt[k]{\left|\frac{3^k}{(k+1)^2}\right| |x^2|^k}$

$= \frac{3}{(\sqrt[k]{k+1})^2} |x|^2 \to 3|x|^2$ as $k \to \infty$ and $3|x|^2 < 1$

$\Rightarrow |x| < \sqrt{\frac{1}{3}}$ $\quad -\sqrt{\frac{1}{3}} < x < \sqrt{\frac{1}{3}}$

$\circ$ Check endpt. As $x = \sqrt{\frac{1}{3}}$, $\sum \frac{3^k}{(k+1)^2} \cdot \left(\sqrt{\frac{1}{3}}\right)^{2k} = \sum \frac{3^k}{(k+1)^2} \cdot \frac{1}{3^k} = \sum \frac{1}{(k+1)^2}$ converges

16. (Section 11.8, Problem 1)

and As $x = -\sqrt{\frac{1}{3}}$ $\sum \frac{3^k}{(k+1)^2} \cdot \left(-\sqrt{\frac{1}{3}}\right)^{2k} = \sum \frac{3^k}{(k+1)^2} \cdot \frac{1}{3^k} = \sum \frac{1}{(k+1)^2}$

$\Rightarrow \boxed{-\sqrt{\frac{1}{3}} \le x \le \sqrt{\frac{1}{3}}}$ or $x \in \left[-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right]$.

$f(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right)$

$= (1 + x + x^2 + x^3 + \cdots + x^n + \cdots)(1 + x + x^2 + x^3 + \cdots + x^n + \cdots)$

$= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$

17. (Section 11.8, Problem 4)

$$f(x)=\ln(1-x) = -\int \frac{dx}{1-x}$$

$$= -\int (1+x+x^2+\cdots+x^n)\,dx$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots - \frac{x^{n+1}}{n+1} + \cdots + C = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^{n+1}}{n+1} \cdots$$

$$\boxed{\text{but } f(0)=\ln(1-0)=0 \Rightarrow C=0}$$

18. (Section 11.8, Problem 7)

$$f(x)=\sec^2 x = (\tan x)'$$

$$= \left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7+\cdots\right)'$$

$$= 1+x^2+\frac{2}{3}x^4+\frac{17}{45}x^6+\cdots$$

19. (Section 11.8, Problem 8)

$$f(x)=\ln\cos x = -\int \tan x\,dx$$

$$= -\int \left[x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7+\cdots\right]dx$$

$$= -\left[\frac{x^2}{2}+\frac{x^4}{12}+\frac{2}{90}x^6+\frac{17}{2520}x^8+\cdots\right] + C$$

$$\boxed{\text{but } f(0)=\ln 1=0 \Rightarrow C=0}$$

20. (Section 11.8, Problem 9)

$$f(x)=x^2\sin x \qquad f^{(9)}(0)=? \quad -72$$

Using Taylor series of $\sin x$ in powers of $x$, we have

$P_9$ of $f(x)$ is $x^2\left(\frac{x}{1}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\right)$

$$= \frac{x^3}{1}-\frac{x^5}{3!}+\frac{x^7}{5!}\boxed{-\frac{x^9}{7!}}$$

$$= \frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(5)}(0)}{5!}x^5+\frac{f^{(7)}(0)}{7!}x^7+\boxed{\frac{f^{(9)}(0)}{9!}}x^9$$

$$\Rightarrow f^{(9)}(0) = -\frac{9!}{7!} = -9\times 8 = -72$$

21. (Section 11.8, Problem 11)

Using $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{(2k+1)}}{(2k+1)!}$

$$\Rightarrow \sin x^2 = \sum_{k=0}^{\infty}(-1)^k\frac{(x^2)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{x^{4k+2}}{(2k+1)!}$$

22. (Section 11.8, Problem 15)

$$f(x)=\frac{2x}{1-x^2} = 2x\left(\frac{1}{1-x^2}\right) = 2x\left(1+x^2+x^4+x^6+\cdots\right)$$

$$= 2x\left(\sum_{k=0}^{\infty}x^{2k}\right) = \sum_{k=0}^{\infty}2x^{2k+1}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

23. (Section 11.8, Problem 19)

$$f(x) = x \cdot \ln(1+x^3)$$

$$= x \cdot \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x^3)^k}{k} = x \cdot \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{3k}}{k}$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{3k+1}}{k}$$