Math 1432, Section 12869
Spring 2014
Homework Assignment 13
Due Date: 4/21/14 in Lab

Name: Sol

ID: ______________________

Instructions

- Print out this file and complete the problems. You must do all the problems!
- If the problem is from the text, the section number and problem number are in parantheses.
- Use a blue or black pen or a pencil (dark).
- Write your solutions in the spaces provided. You must show work in order receive credit for a problem.
- Remember that your homework must be complete, neatly written and stapled.
- Submit the completed assignment to your Teaching Assistant in lab on the due date.
- If you do not do all of the problems, then your recitation quiz from the previous Friday will automatically become a ZERO.

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1. (Section 11.4, Problem 2)

$$\frac{1}{4}-\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots=\sum_{k=2}^{\infty}\frac{(-1)^k}{2k}$$

(a). $\sum_{k=2}^{\infty}\left|\frac{(-1)^k}{2k}\right| = \sum_{k=2}^{\infty}\frac{1}{2k}$ diverges by Limit Comparison Test and compare with $\sum\frac{1}{k}$

Thus $\sum_{k=2}^{\infty}\frac{(-1)^k}{2k}$ isn't abs. convergent

(b) Let $a_k=\frac{1}{2k}$. by Alternating Series Test.

Since $a_k\to 0$, then $\sum_{k=2}^{\infty}(-1)^k\frac{1}{2k}$ converges

So it is Conditional convergent.

2. (Section 11.4, Problem 3)

$$\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{k+1}$$

(Basic Divergence Test)

(a) $\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{k}{k+1}\right| = \sum_{k=1}^{\infty}\frac{k}{k+1}$ diverges since $\frac{k}{k+1}\to 1\neq 0$ as $k\to\infty$

So $\sum(-1)^{k+1}\frac{k}{k+1}$ isn't absolutely convergent.

(b) since $\frac{k}{k+1}\to 1\neq 0$ as $k\to\infty$, By Basic Divergence Test

So $\sum(-1)^{k+1}\frac{k}{k+1}$ isn't conditional convergent.

3. (Section 11.4, Problem 5)

$\sum(-1)^k\frac{\ln k}{k}$ (a), $\sum\left|(-1)^k\frac{\ln k}{k}\right| = \sum\frac{\ln k}{k}$ diverges

by Basic Comparison Test and compare with $\sum\frac{1}{k}$

so it isn't abs. convergent.

(b) since $\frac{\ln k}{k}\to 0$ as $k\to\infty$. so, by Alternating Series Test, It is Conditional convergent.

4. (Section 11.4, Problem 8)

$\sum\frac{k^3}{2^k}$. (a) $\sum\left|\frac{k^3}{2^k}\right| = \sum\frac{k^3}{2^k}$. let $a_k=\frac{k^3}{2^k}$.

by Root test, $\sqrt[k]{a_k}=\frac{\sqrt[k]{k^3}}{2}=\frac{(\sqrt[k]{k})^3}{2}\to\frac{1}{2}$ as $k\to\infty$

($\sqrt[k]{k}\to 1$ as $k\to\infty$)

$\Rightarrow$ It is absolutely convergent!

5. (Section 11.4, Problem 9)

$\sum(-1)^k \frac{1}{2k+1}$ (a) $\sum\left|(-1)^k\frac{1}{2k+1}\right| = \sum\frac{1}{2k+1}$ diverges by Limit Comparison Test and compare it with $\sum\frac{1}{k}$.

so it isn't abs. convergent!

(b) since $\frac{1}{2k+1} \to 0$, By Alternating Series Test,

$\sum(-1)^k\frac{1}{2k+1}$ Converges, so it is conditional convergent!

6. (Section 11.4, Problem 12)

$\sum \sin(\frac{k\pi}{4})$, (a) $\sum\left|\sin(\frac{k\pi}{4})\right|$ diverges

since $\left|\sin(\frac{k\pi}{4})\right| \not\to 0$ as $k\to\infty$ (by Basic Comparison Test)

so $\sum \sin(\frac{k\pi}{4})$ isn't abs. convergent!

(b) $\sum \sin(\frac{k\pi}{4})$ diverges since $\sin(\frac{k\pi}{4}) \not\to 0$ as $k\to\infty$

so $\sum\sin(\frac{k\pi}{4})$ isn't conditional convergent;

7. (Section 11.4, Problem 13)

$\sum(-1)^k(\sqrt{k+1}-\sqrt{k})$ (a) $\sum\left|(-1)^k(\sqrt{k+1}-\sqrt{k})\right| = \sum \sqrt{k+1}-\sqrt{k}$.

$\overset{*}{=} \sum \sqrt{k+1}-\sqrt{k}\cdot\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}} = \sum\frac{1}{\sqrt{k+1}+\sqrt{k}}$ diverges

by Limit Comparison Test, and compare it with $\sum\frac{1}{\sqrt{k}}$.

So it is not abs. convergent;

(b) Since $\sqrt{k+1}-\sqrt{k} \overset{*}{=} \frac{1}{\sqrt{k+1}+\sqrt{k}} \to 0$ as $k\to\infty$

So, by Alternating Series Test, $\sum(-1)^k(\sqrt{k+1}-\sqrt{k})$ converges and it is conditional convergent!

8. (Section 11.4, Problem 16)

$\sum\frac{(-1)^k}{\sqrt{k(k+1)}}$ (a) $\sum\left|\frac{(-1)^k}{\sqrt{k(k+1)}}\right| = \sum\frac{1}{\sqrt{k(k+1)}} \approx \sum\frac{1}{\sqrt{k^2}} = \sum\frac{1}{k}$

diverges by limit Comparison Test.

So it is NOT abs. convergent;

(b) since $\frac{1}{\sqrt{k(k+1)}} \to 0$ as $k\to\infty$, by Alternating Series Test, $\sum(-1)^k\frac{1}{\sqrt{k(k+1)}}$ converges, and is conditional convergent!

9. (Section 11.4, Problem 17)

$\sum(-1)^k\frac{k}{2^k}$ (a) $\sum\left|(-1)^k\frac{k}{2^k}\right| = \sum\frac{k}{2^k}$ converges by Root test: Let $a_k = \frac{k}{2^k}$, $\sqrt[k]{a_k} = \frac{\sqrt[k]{k}}{2} \to \frac{1}{2} < 1$ as $k\to\infty$

So $\sum(-1)^k\frac{k}{2^k}$ is abs. convergent!

(b) Since $\sum(-1)^k\frac{k}{2^k}$ is abs. convergent,

so it is conditional convergent, too.

10. (Section 11.4, Problem 20)

$\sum(-1)^k\frac{k+2}{k^2+k}$ (a) $\sum\left|(-1)^k\frac{k+2}{k^2+k}\right| = \sum\frac{k+2}{k^2+k} \approx \sum\frac{k}{k^2} = \sum\frac{1}{k}$

diverges by Limit Comparison Test.

So $\sum(-1)^k\frac{k+2}{k^2+k}$ is NOT abs. convergent;

(b) since $\frac{k+2}{k^2+k} \to 0$ as $k\to\infty$. Then, by Alternating Series Test, $\sum(-1)^k\frac{k+2}{k^2+k}$ converges,

so it is conditional convergent;

$$\cos \pi k = \cos\pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \cdots$$
$$(\star) \qquad \cos\pi = -1,\ \cos 2\pi = 1,\ \cos 3\pi = -1,\ \cos 4\pi = 1 \Rightarrow (-1)^k$$

11. (Section 11.4, Problem 25)

$\sum (-1)^k k e^{-k} = \sum (-1)^k \frac{k}{e^k}$ (a) $\sum \left|(-1)^k \frac{k}{e^k}\right| = \sum \frac{k}{e^k}$ converges

by Root test, since, let $a_k = \frac{k}{e^k}$, $\sqrt[k]{a_k} = \frac{\sqrt[k]{k}}{e} \to \frac{1}{e} < 1$ as $k \to \infty$.

So it is abs. convergent!

(b) by (a). since $\sum (-1)^k k e^{-k}$ is abs. convergent,

so it is conditional convergent, too.

12. (Section 11.4, Problem 26)

$\sum \frac{\cos \pi k}{k} = \sum \frac{(-1)^k}{k}$ (a) $\sum \left|\frac{(-1)^k}{k}\right| = \sum \frac{1}{k^1}$ diverges

by P-series Test for $p \le 1$, so $\sum \frac{(-1)^k}{k}$ is NOT

abs. convergent;

(b) since $\frac{1}{k} \to 0$ as $k \to \infty$. Then, by Alternating Series

Test, $\sum (-1)^k \frac{1}{k}$ converges. so it is ~~conditional convergent!~~

13. (Section 11.4, Problem 27)

$\sum (-1)^k \frac{\cos \pi k}{k} \overset{(\star)}{=} \sum \frac{(-1)^{2k}}{k} = \sum \frac{1}{k}$

(a) $\sum \left|\frac{1}{k}\right| = \sum \frac{1}{k}$ diverges by p-series test.

So it is NOT abs. convergent!

(c) $\sum \frac{1}{k}$ diverges by p-series test,

So it is NOT conditional convergent!

Taylor Polynomials in x:

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + R_n(x)$$

~~where remainder~~ $R_n(x) = \frac{1}{n!}\int f^{(n+1)}(t)(x-t)^n dt$,

14. (Section 11.5, Problem 1)

$f(x) = x - \cos x$, $f(0) = -1$

$f'(x) = 1 + \sin x$ $\quad f'(0) = 1$ $\quad \Rightarrow P_4 = -1 + \frac{1}{1!}x + \frac{1}{2!}x^2$

$f''(x) = \cos x$ $\quad f''(0) = 1$ $\quad + 0x^3 + \frac{-1}{4!}x^4$

$f'''(x) = -\sin x$ $\quad f'''(0) = 0$ $\quad = \boxed{-1 + x + \frac{x^2}{2} - \frac{1}{4!}x^4}$.

$f^{(4)}(x) = -\cos x$ $\quad f^{(4)}(0) = -1$

15. (Section 11.5, Problem 2)

$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$, $f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\quad f''(0) = -\frac{1}{4}$

$f'''(x) = +\frac{3}{8}(1+x)^{-\frac{5}{2}}$ $\quad f'''(0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-\frac{7}{2}}$ $\quad f^{(4)}(0) = -\frac{15}{16}$

$\Rightarrow P_4 = 1 + \frac{\frac{1}{2}}{1!}x + \frac{-\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3 + \frac{-\frac{15}{16}}{4!}x^4$

$= \boxed{1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{3}{8\cdot 3!}x^3 - \frac{15}{16\cdot 4!}x^4}$

16. (Section 11.5, Problem 3)

$f(x) = \ln \cos x$ $\quad \Rightarrow f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{\cos x}\cdot(-\sin x) = -\tan x$ $\quad \Rightarrow f'(0) = 0$

$f''(x) = -\sec^2 x = -(\sec x)^2$ $\quad \Rightarrow f''(0) = -1$

$f'''(x) = -2(\sec x)\cdot \sec x \tan x = -2(\sec x)^2 \tan x$ $\quad \Rightarrow f'''(0) = 0$

$f^{(4)}(x) = -4(\sec x)\cdot \tan x \cdot \sec x \tan x - 2(\sec x)^2 \sec x$

$= -4(\sec x)^2 \tan^2 x - 2\sec^3 x$ $\quad \Rightarrow f^{(4)}(0) = -2$.

$\Rightarrow P_4 = 0 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-2}{4!}x^4 = \boxed{-\frac{1}{2}x^2 - \frac{2}{24}x^4}$.

P_5 $\quad P_5 = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{0}{4!}x^4 + \frac{-60}{5!}x^5$

17. (Section 11.5, Problem 8)

$= \boxed{x - \frac{1}{2}x^5}$

$f(x) = x\cos x^2$, $\quad f(0) = 0$

$f'(x) = \cos x^2 + x\cdot(2x)\cdot(-\sin x^2)$, $\quad f'(0) = 1$

$f''(x) = 2x\cdot(-\sin x^2) + 4x(-\sin x^2) - 4x^3(\cos x^2)$, $\quad f''(0) = 0$

$= -6x\cdot\sin x^2 - 4x^3\cdot\cos x^2$

$f'''(x) = -6\sin x^2 - 6x\cdot(2x)(\cos x^2) - 12x^2\cdot\cos x^2 + 4x^3\cdot 2x\sin x^2$

$= (8x^4 - 6)\sin x^2 - 24x^2\cos x^2$ $\quad f'''(0) = 0$

$f^{(4)}(x) = (32x^3)\sin x^2 + 2x(8x^4-6)\cos x^2 - 48x\cos x^2 + 96x\sin x^2$

$= (32x^3 + 96x)\sin x^2 + [16x^5 - 60x]\cos x^2$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = (96x^2 + 96)\sin x^2 + 2x(32x^3 + 96x)\cos x^2$

$+ [90x^4 - 60]\cos x^2 - 2x[16x^5 - 60x]\sin x^2$

$f^{(5)}(0) = -60$

18. (Section 11.5, Problem 11)

$f(x) = e^{-x}$, $f(0) = 1$

$f'(x) = (-1)e^{-x}$, $f'(0) = (-1)$

$f''(x) = (-1)^2 e^{-x}$, $f''(0) = (-1)^2$

$f'''(x) = (-1)^3 e^{-x}$, $f'''(0) = (-1)^3$

$f^{(4)}(x) = (-1)^4 e^{-x}$

$\vdots$

$f^{(n)}(x) = (-1)^n e^{-x}$, $f^{(n)}(0) = (-1)^n$

$\Rightarrow P_n = \boxed{1 + \frac{-1}{1!}x + \frac{(-1)^2}{2!}x^2 + \frac{(-1)^3}{3!}x^3 + \cdots + \frac{(-1)^n}{n!}x^n}$

19. (Section 11.5, Problem 12)

$f(x) = \sinh x$, $\quad f(0) = 0$

$f'(x) = \cosh x$, $\quad f'(0) = 1$

$f''(x) = \sinh x$, $\quad f''(0) = 0$

$f'''(x) = \cosh x$, $\quad f'''(0) = 1$.

$\vdots$

$f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd} \end{cases}$

$\boxed{P_n = \frac{1}{1!}x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots + \frac{1}{(2m-1)!}x^{2m-1}}$

Note.

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

20. (Section 11.5, Problem 15)

$f(x) = e^{rx}$, $f(0) = 1$

$f'(x) = re^{rx}$, $f'(0) = r$

$f''(x) = r^2 e^{rx}$, $f''(0) = r^2$

$f'''(x) = r^3 e^{rx}$, $f'''(0) = r^3$

$\vdots \qquad \vdots$

$f^{(n)}(x) = r^n e^{rx}$, $f^{(n)}(0) = r^n$

$\boxed{P_n = 1 + \frac{r}{1!}x + \frac{r^2}{2!}x^2 + \frac{r^3}{3!}x^3 + \cdots + \frac{r^n}{n!}x^n}$

21. (Section 11.5, Problem 16)

$f(x) = \cos bx$ $\quad f(0) = 1$

$f'(x) = b\cdot(-1)\sin bx$, $\quad f'(0) = 0$

$f''(x) = b^2\cdot(-1)\cos bx$, $\quad f''(0) = (-1)b^2$

$f'''(x) = b^3\cdot(-1)^2\sin bx$, $\quad f'''(0) = 0$

$f^{(4)}(x) = b^4\cdot(-1)^2\cos bx$, $f^{(4)}(0) = (-1)^2 b^4$

$f^{(5)}(x) = b^5(-1)^3\sin bx$, $f^{(5)}(0) = 0$

$f^{(6)}(x) = b^6(-1)^3\cos bx$ $\quad f^{(6)}(0) = (-1)^3 b^6$

$f^{(k)}(0) = \begin{cases} (-1)^{\frac{k}{2}} b^k & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$

$P_n = 1 + \frac{(-1)b^2}{2!}x^2 + \frac{(-1)^2 b^4}{4!}x^4 + \frac{(-1)^3 b^6}{6!}x^6 + \cdots + \frac{(-1)^{\frac{m}{2}} b^m}{m!}x^m$

(if n is odd, $m = n-1$; if n is even $m = n$)

22. (Section 11.5, Problem 17)

$|f^{(n)}(x)| \le 1$ is given $\forall n$ and $\forall x$. — $(*)$

and the REMAINDER for P_n is

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$

Now, we use $P_5(\frac{1}{2})$, (ie $\begin{cases} x = \frac{1}{2} \\ n = 5 \end{cases}$), we have

$|R_5(\frac{1}{2})| = \left|\frac{f^{(5+1)}(c)}{(5+1)!}\cdot\left(\frac{1}{2}\right)^{5+1}\right| \overset{(*)}{\le} \frac{1}{6!}\cdot\left(\frac{1}{2}\right)^6 =$

$f(x)=\ln(1+x)$, $f(0)=0$

$f'(x)=\frac{1}{1+x}=(1+x)^{-1}$, $f'(0)=1$

$f''(x)=\frac{-1}{(1+x)^2}=-1(1+x)^{-2}$, $f''(0)=(-1)$

$f'''(x)=(-1)(-2)(1+x)^{-3}$ $f'''(0)=(-1)(-2)=(-1)^2 2!$

$f^{(4)}=(-1)(-2)(-3)(1+x)^{-4}$

$f^{(4)}(0)=(-1)^3\cdot 3!$

$\vdots$

$f^{(n)}(0)=(-1)^{n-1}(n-1)!$

23. (Section 11.5, Problem 19)

(*) $|f^{(n)}(x)|\le 1$ is given, we use $P_n(2)$ such that $|R_n(2)|\le 0.001$

$$\Rightarrow |R_n(2)|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}2^{n+1}\right|\le \frac{1}{(n+1)!}2^{n+1}\le 0.001$$

Find n such that $\frac{2^{n+1}}{(n+1)!}\le 0.001$

$n=5$, $\frac{64}{720}\ge 0.001$; $n=6$, $\frac{128}{5040}=0.025$; $n=7$, $\frac{256}{40320}=0.0063$

$n=8$, $\frac{512}{362880}=0.0014\ge 0.001$

$n=9$, $\frac{1024}{3628800}=0.0003\le 0.001$

$n=9$

24. (Section 11.5, Problem 22)

$|f^{(n)}(x)|\le 3$ is given (*)

We use $P_n(2)$ such that $f(2)$ has three decimal place accuracy. $\Rightarrow$

$$|R_n(2)|\le 0.0005 \Rightarrow |R_n(2)|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}2^{n+1}\right|\le\frac{3\cdot 2^{n+1}}{(n+1)!}\le 0.0001$$

Try $n=9$ $\frac{3\cdot 2^{10}}{10!}=0.0009\ge 0.0005$, $n=10$ $\frac{3\cdot 2^{11}}{11!}=0.000163\le 0.0005$

$n=10$

25. (Section 11.5, Problem 25)

Estimate $\sqrt{e}=e^{\frac{1}{2}}$.

Let $f(x)=e^x$. To find the least integer n for which $P_n(\frac{1}{2})$ approximates $f(\frac{1}{2})$ to within 0.01, it is sufficient to find n s.t. $|R_n(\frac{1}{2})|\le 0.01$

$c\in(0,\frac{1}{2})$

Since $f^{(n)}(x)=e^x$, then $|R_n(\frac{1}{2})|=\left|\frac{e^c}{(n+1)!}(\frac{1}{2})^{n+1}\right|\le e^{\frac{1}{2}}\frac{1}{(n+1)!}(\frac{1}{2})^{n+1}$

$e^{\frac{1}{2}}\le 2$

$\le 2\cdot\frac{1}{(n+1)!}\cdot(\frac{1}{2})^{n+1}\le 0.01$

Try $n=2$ $2\cdot\frac{1}{3!}\cdot(\frac{1}{2})^3=\frac{1}{6\cdot 4}=\frac{1}{24}>0.01$

$\boxed{n=3}$ $2\cdot\frac{1}{4!}(\frac{1}{2})^4=\frac{1}{48\cdot 4}=0.005\le 0.01$

26. (Section 11.5, Problem 29)

Estimate $\ln 1.2$. Let $f(x)=\ln(1+x)$, and $P_n(x)=$

$$P_n(x)=x+\frac{(-1)}{2!}x^2+\frac{(-1)^2 2!}{3!}x^3+\frac{(-1)^3 3!}{4!}x^4+\cdots+\frac{(-1)^{n-1}(n-1)!}{n!}x^n$$

$$=x-\frac{1}{2}x^2+\frac{(-1)^2}{3}x^3+\frac{(-1)^3}{4}x^4+\cdots+\frac{(-1)^{n-1}}{n}x^n$$

Find n s.t. $P_n(\frac{1}{5})$ approximates $f(\frac{1}{5})=\ln(1+\frac{1}{5})$ within 0.01

$$\Rightarrow |R_n(\tfrac{1}{5})|\le 0.01\Rightarrow\left|\frac{(-1)^n n!(1+c)^{-(n+1)}}{(n+1)!}(\tfrac{1}{5})^{n+1}\right|\le 0.01$$

Try $\boxed{n=2}$ $\Rightarrow$ $\boxed{\text{let } c=0}$ $\frac{1}{3}\cdot(\frac{1}{5})^3\le 0.01$ $\Rightarrow P_2(\frac{1}{2})=\frac{1}{2}-\frac{1}{2}(\frac{1}{2})^2=0.18$

27. (Section 11.5, Problem 33)

$f(x)=e^{2x}$, $n=4$, $f^{(n)}(x)=2^n e^{2x}$

$$R_4(x)=\frac{2^{(4+1)}e^{2\cdot c}}{(4+1)!}x^{4+1}$$

$$=\frac{2^5 e^{2c}}{5!}x^5 \text{ for } c\in(0,x).$$

28. (Section 11.5, Problem 35)

$f(x)=\cos 2x$ $n=4$

$$R_4(x)=\frac{f^{(5)}(c)}{5!}x^5=\frac{2^5(-1)^3\sin 2c}{5!}x^5,\ c\in(0,x)$$

$f'(x)=2(-1)\sin 2x$, $f^{(4)}(x)=2^4(-1)^2\cos 2x$

$f''(x)=2^2(-1)\cos 2x$ $f^{(5)}(x)=2^5(-1)^3\sin 2x$

$\cancel{f'''(x)=2^3(-1)^2\sin 2x}$

So we have to find $P_3(\frac{1}{2})$, which is

$$1+\frac{1}{1!}\frac{1}{2}+\frac{1}{2!}(\tfrac{1}{2})^2+\frac{1}{3!}\cdot(\tfrac{1}{2})^3=\frac{79}{48}$$

29. (Section 11.5, Problem 43)

$f(x) = \frac{1}{1-x} = (1-x)^{-1}$

$f'(x) = (-1)(1-x)^{-2}$

$f''(x) = (-1)(-2)(1-x)^{-3} = (-1)^2 2!\,(1-x)^{-3}$

$f'''(x) = (-1)^3 3!\,(1-x)^{-4}$

$\vdots$

$f^{(n)}(x) = (-1)^n n!\,(1-x)^{-(n+1)}$

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$

$= \frac{(-1)^{n+1}(n+1)!\,(1-c)^{-(n+2)}}{(n+1)!} x^{n+1}$

$= (-1)^{n+1} (1-c)^{-(n+2)} x^{n+1}$

$= (1-c)^{-(n+2)} x^{n+1}$

30. (Section 11.5, Problem 44)

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{(1+x)} = (1+x)^{-1}$

$f''(x) = (-1)\cdot(1+x)^{-2}$

$f'''(x) = (-1)^2 2!\,(1+x)^{-3}$

$f^{(4)}(x) = (-1)^3 3!\,(1+x)^{-4}$

$\vdots$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$

$= \frac{(-1)^{(n+1)-1}[(n+1)-1]!\,(1+c)^{-(n+1)}}{(n+1)!} x^{n+1}$

$= \frac{(-1)^n n!\,(1+c)^{-(n+1)}}{(n+1)!} x^{n+1}$

$= \frac{(-1)^n (1+c)^{-(n+1)}}{n+1} x^{n+1}$

31. (Section 11.5, Problem 45)

(a) $|R_n(0.5)| \le 0.01 \Rightarrow \left| \frac{(-1)^n (1+c)^{-(n+1)}}{n+1} \left(\frac{1}{2}\right)^{n+1} \right| \le 0.01$

$c=0$

$\Rightarrow \frac{1}{n+1}\cdot\left(\frac{1}{2}\right)^{n+1} \le 0.01$, Try $n=2$, $\frac{1}{3}\cdot\left(\frac{1}{2}\right)^3 = \frac{1}{3}\cdot\frac{1}{8} = \frac{1}{24} > 0.01$

$n=3$ $\frac{1}{4}\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{4}\cdot\frac{1}{16} = \frac{1}{64} = 0.015 > 0.01$, $\boxed{n=4}$, $\frac{1}{5}\cdot\left(\frac{1}{2}\right)^5 = \frac{1}{5\cdot 32} =$ $0.006 \le 0.01$

$n \ge 4$

(b) $|R_n(0.3)| \le 0.01 \Rightarrow \frac{1}{n+1}(0.3)^{n+1} \le 0.01$

Try $\boxed{n=2}$, $\frac{1}{3}\cdot\left(\frac{3}{10}\right)^3 = 0.009 \le 0.01$ $\Rightarrow n \ge 2$

(c) $|R_n(1)| \le 0.001 \Rightarrow \frac{1}{n+1}(1)^{n+1} = \frac{1}{n+1} \le 0.001$. Try $\boxed{n=999}$

$n \ge 999$

32. (Section 11.6, Problem 1)

$x^{\frac{1}{2}}$

$f(x) = \sqrt{x}$; $a=4$, $n=3$. $f(4) = 2$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f'(4) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$f''(x) = \left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$ $f''(4) = -\frac{1}{4}\cdot\frac{1}{2^3} = -\frac{1}{32}$

$f'''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$, $f'''(4) = \frac{3}{8}\cdot\frac{1}{2^5} = \frac{3}{8\cdot 32}$

$P_3 = \boxed{2 + \frac{1}{4}(x-4) + \frac{-1}{32}\cdot\frac{1}{2!}(x-4)^2 + \frac{3}{8\cdot 32}\cdot\frac{1}{3!}(x-4)^3}$

Lagrange formula for $R_3(x)$.

$f^{(4)}(x) = \frac{3}{8}\cdot\left(-\frac{5}{2}\right)x^{-\frac{7}{2}}$

$R_3(x) = \frac{-\frac{15}{16}c^{-\frac{7}{2}}}{4!}(x-4)^4$

c is between 4 and x.

33. (Section 11.6, Problem 2)

$f(x) = \cos x$, $a = \frac{\pi}{3}$, $n=4$ $\rightarrow$ $f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x) = -\sin x$, $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$, $f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$f'''(x) = \sin x$, $f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$f^{(4)}(x) = \cos x$, $f^{(4)}\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$R_4(x)$:

$f^{(5)}(x) = -\sin x$.

$R_4(x) = \frac{-\sin c}{5!}\left(x-\frac{\pi}{3}\right)^5$

which c is between $\frac{\pi}{3}$ and x.

$P_4 = \frac{1}{2} + \frac{-\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) + \frac{-1}{2}\cdot\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{2}\left(x-\frac{\pi}{3}\right)^4$

34. (Section 11.6, Problem 4) $a=1$, $n=5$

$f(x) = \ln x$, $f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$, $f'(1) = 1$

$f''(x) = -1\cdot x^{-2}$, $f''(1) = -1$

$f'''(x) = (-1)(-2)x^{-3}$, $f'''(1) = (-1)\cdot(-2) = 2!$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$, $f^{(4)}(1) = -3!$

$f^{(5)}(x) = (-1)(-2)(-3)(-4)x^{-5}$, $f^{(4)}(1) = 4!$

$R_5(x)$:

$f^{(6)}(x) = -5!\,x^{-6}$

$R_5(x) = \frac{-5!\,c^{-6}}{6!}(x-1)^6$

c is between 1 and x.

$P_5 = 0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \frac{-3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5$

$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$

Another method: $g(x)=\frac{1}{x}=\frac{1}{1-1+x}=\frac{1}{1+(x-1)}$ (Geometric series)

$\cancel{=\sum(-1)^n(x+1)}$ $=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

$(0!=1)$

Let $a_k=\frac{(-1)^{k+1}}{(2k+1)!}(x-\frac{\pi}{2})^{2k+1}$, $\frac{a_{k+1}}{a_k}=\frac{(-1)}{(2k+3)(2k+2)}(x-\frac{\pi}{2})^2$

Then $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=0<1 \Rightarrow \boxed{x\in(-\infty,\infty)}$

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35. (Section 11.6, Problem 10) $g(x)=x^{-1}$ in powers of $x-1$

$g(x)=x^{-1}$, $g(1)=1$

$g'(x)=-1\cdot x^{-2}=-1\cdot x^{-2}$, $g'(1)=-1$

$g''(x)=(-1)(-2)x^{-3}=(-1)^2 2!\,x^{-3}$, $g''(1)=(-1)^2 2!$

$g'''(x)=(-1)(-2)(-3)x^{-4}=(-1)^3 3!\,x^{-4}$; $g'''(1)=(-1)^3 3!$

$g^{(4)}(x)=(-1)(-2)(-3)(-4)x^{-5}=(-1)^4 4!\,x^{-5}$, $g^{(4)}(1)=(-1)^4 4!$

$\vdots$

$g^{(n)}(x)=(-1)^n n!\,x^{-(n+1)}$, $g^{(n)}(1)=(-1)^n\cdot n!$

$\Rightarrow \boxed{g(x)=\sum_{n=0}^{\infty}\frac{(-1)^n n!}{n!}(x-1)^n}$

$=\boxed{\sum_{n=0}^{\infty}(-1)^n(x-1)^n}$

① Let $a_n=(-1)^n(x-1)^n$.

It is valid $\Leftrightarrow$ $\lim_{n\to\infty}\sqrt[n]{|a_n|}<1$ (Root test)

$\Rightarrow |x-1|<1 \Rightarrow \boxed{x\in(0,2)}$

(x=0 and x=2 are not valid since g(x) DNE as x=0 and $g(2)\neq\sum_{n=0}^{\infty}(-1)^n(1)^n$)

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36. (Section 11.6, Problem 13) in the powers of $x+1$

$g(x)=e^{-4x}$, $g(-1)=e^4$

$g'(x)=-4e^{-4x}$, $g'(-1)=-4\cdot e^4$

$g''(x)=(-4)^2e^{-4x}$, $g''(-1)=(-4)^2e^4$

$g'''(x)=(-4)^3e^{-4x}$, $g'''(-1)=(-4)^3e^4$

$\vdots$

$g^{(n)}(x)=(-4)^ne^{-4x}$, $g^{(n)}(-1)=(-4)^ne^4$

Let $a_n=\frac{(-4)^ne^4}{n!}(x+1)^n$

This series valid $\Leftrightarrow \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}<1$.

For a fixed x

$\Rightarrow \lim_{n\to\infty}\left|\frac{-4}{n+1}(x+1)\right|=0<1$, $\forall x \Rightarrow \boxed{x\in(-\infty,\infty)}$

$\boxed{g(x)=e^4+\frac{-4\cdot e^4}{1!}(x+1)+\frac{(-4)^2e^4}{2!}(x+1)^2+\cdots+\frac{(-4)^ne^4}{n!}(x+1)^n=\sum_{n=0}^{\infty}\frac{(-4)^ne^4}{n!}(x+1)^n}$

---

37. (Section 11.6, Problem 17) In powers of $x-\pi$.

$g(x)=\cos x$, $g(\pi)=-1=(-1)^1$

$g'(x)=-\sin x$, $g'(\pi)=0$

$g''(x)=(-1)\cdot\cos x$, $g''(\pi)=(-1)(-1)=(-1)^2$

$g'''(x)=(-1)^2\sin x$, $g'''(\pi)=0$

$g^{(4)}(x)=(-1)^2\cos x$, $g^{(4)}(\pi)=(-1)^2(-1)=(-1)^3$

$g^{(5)}(x)=(-1)^3\sin x$, $g^{(5)}(\pi)=0$

$g^{(6)}(x)=(-1)^3\cos x$, $g^{(6)}(\pi)=(-1)^4$

$g^{(n)}(\pi)=\begin{cases}0 & n\text{ is odd}\\ (-1)^{\frac{n}{2}+1} & n\text{ is even}\end{cases}$

Let $a_k=\frac{(-1)^{k+1}}{(2k)!}(x-\pi)^{2k}$. It is valid $\Leftrightarrow \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|<1$

Then $\frac{a_{k+1}}{a_k}=\frac{-1}{(2k+2)(2k+1)}(x-\pi)^2$ $\Rightarrow \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|\to 0<1 \Rightarrow \boxed{x\in(-\infty,\infty)}$

(even number) $n=2k$

$\boxed{g(x)=(-1)^1+0(x-\pi)+\frac{(-1)^2}{2!}(x-\pi)^2+0(x-\pi)^3+\cdots+\frac{(-1)^{\frac{n}{2}+1}}{n!}(x-\pi)^n=\sum_{k=0}^{\infty}\frac{(-1)^{\frac{2k}{2}+1}}{(2k)!}(x-\pi)^{2k}=\sum\frac{(-1)^{k+1}}{(2k)!}(x-\pi)^{2k}}$

---

38. (Section 11.6, Problem 18) in powers of $x-\frac{\pi}{2}$.

$g(x)=\cos x$, $g(\frac{\pi}{2})=0$

$g'(x)=-\sin x$, $g'(\frac{\pi}{2})=-1$

$g''(x)=(-1)\cos x$, $g''(\frac{\pi}{2})=0$

$g'''(x)=(-1)^2\sin x$, $g'''(\frac{\pi}{2})=(-1)^2$

$g^{(4)}(x)=(-1)^2\cos x$, $g^{(4)}(\frac{\pi}{2})=0$

$g^{(5)}(x)=(-1)^3\sin x$, $g^{(5)}(\frac{\pi}{2})=(-1)^3$

$g^{(6)}(x)=(-1)^3\cos x$

$\Rightarrow g^{(n)}(\frac{\pi}{2})=\begin{cases}0 & \text{if } n\text{ is even}\\ (-1)^{\frac{n+1}{2}} & \text{if } n\text{ is odd}\end{cases}$ let $n=2k+1$

$\Rightarrow \boxed{g(x)=\sum_{k=0}^{\infty}\frac{(-1)^{\frac{2k+1+1}{2}}}{(2k+1)!}(x-\frac{\pi}{2})^{2k+1}}$

$=\boxed{\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}(x-\frac{\pi}{2})^{2k+1}}$

---

39. (Section 11.6, Problem 21) In powers of $x-1$

$g(x)=\ln(1+2x)$, $g(1)=\ln 3$

$g'(x)=\frac{2}{1+2x}=2\cdot(1+2x)^{-1}$, $g'(1)=2\cdot\frac{1}{3}$

$g''(x)=2^2\cdot(-1)(1+2x)^{-2}$, $g''(1)=2^2(-1)\cdot(\frac{1}{3})^2$

$g'''(x)=2^3(-1)(-2)(1+2x)^{-3}=2^3\cdot(-1)^2\cdot 2!\,(1+2x)^{-3}$, $\Rightarrow g'''(1)=\frac{2^3(-1)^2 2!}{3^3}$

$g^{(4)}(x)=2^4(-1)(-2)(-3)(1+2x)^{-4}=2^4(-1)^3 3!\,(1+2x)^{-4}$

$\vdots$

$g^{(n)}(x)=2^n(-1)^{n-1}(n-1)!\,(1+2x)^{-n} \Rightarrow g^{(n)}(1)=\frac{2^n(-1)^{n-1}(n-1)!}{3^n}$

$\boxed{g(x)=\ln 3+\sum_{n=1}^{\infty}\frac{2^n(-1)^{n-1}(n-1)!}{3^n}\cdot\frac{1}{n!}(x-1)^n}$

$\boxed{=\ln 3+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{3}\right)^n\cdot\frac{1}{n}(x-1)^n}$

39. $g(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\left(\frac{2}{3}\right)^n(x-1)^n+\ln 3$

let $a_n=\frac{(-1)^{n-1}}{n}\cdot\left(\frac{2}{3}\right)^n\cdot(x-1)^n$.

$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}\cdot\frac{2}{3}\cdot(x-1)=\frac{2}{3}(x-1)$.

(Root test)

$\frac{2}{3}(x-1)<1 \Rightarrow (x-1)<\frac{3}{2} \Rightarrow -\frac{3}{2}<x-1<\frac{3}{2}$

$\Rightarrow \boxed{-\frac{1}{2}<x<\frac{5}{2}}$

as $x=-\frac{1}{2}$. $g(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{2}{3}\right)^n\left(-\frac{3}{2}\right)^n$. div.

as $x=\frac{5}{2}$ $g(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{2}{3}\right)^n\left(\frac{3}{2}\right)^n$ div.