Math 1432, Section 12869
Spring 2014
Homework Assignment 12
Due Date: 4/14/14 in Lab

Name: Sol.

ID:

Instructions

- Print out this file and complete the problems. You must do all the problems!
- If the problem is from the text, the section number and problem number are in parantheses.
- Use a blue or black pen or a pencil (dark).
- Write your solutions in the spaces provided. You must show work in order receive credit for a problem.
- Remember that your homework must be complete, neatly written and stapled.
- Submit the completed assignment to your Teaching Assistant in lab on the due date.
- If you do not do all of the problems, then your recitation quiz from the previous Friday will automatically become a ZERO.

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1. (Section 11.1, Problem 27)

$$\sum_{k=1}^{\infty}\frac{1}{k(k+3)} = \sum_{k=1}^{\infty}\frac{\frac{1}{3}}{k} - \frac{\frac{1}{3}}{k+3}$$

$$= \frac{1}{3}\sum_{k=1}^{\infty}\frac{1}{k} - \frac{1}{k+3}$$

$$= \frac{1}{3}\Big[\underbrace{\left(\frac{1}{1}-\frac{1}{4}\right)}_{k=1} + \underbrace{\left(\frac{1}{2}-\frac{1}{5}\right)}_{k=2} + \underbrace{\left(\frac{1}{3}-\frac{1}{6}\right)}_{k=3} + \underbrace{\left(\frac{1}{4}-\frac{1}{7}\right)}_{k=4} + \underbrace{\left(\frac{1}{5}-\frac{1}{8}\right)}_{k=5} + \underbrace{\left(\frac{1}{6}-\frac{1}{9}\right)}_{k=6} + \cdots\Big]$$

since $\frac{1}{k} \to 0$, $\frac{1}{k+3} \to 0$ as $k \to \infty$

$$= \frac{1}{3}\left[1 + \frac{1}{2} + \frac{1}{3} + 0\right] = \frac{1}{3}\left[\frac{11}{6}\right] = \frac{11}{18}$$

$\frac{11}{18}$

2. (Section 11.1, Problem 29)

$$\sum_{k=0}^{\infty}\frac{3}{10^k} = \frac{3}{10^0} + \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$$

↑ Geometric (see professor's slide (4/7))

It is a series with first term $\frac{3}{10^0} = 3$ and common ratio $\left|\frac{1}{10}\right|$ which is less than 1.

$$\Rightarrow \sum_{k=0}^{\infty}\frac{3}{10^k} = \frac{\frac{3}{10^0}}{1-\frac{1}{10}} = 3\cdot\frac{10}{9} = \frac{10}{3}$$

3. (Section 11.1, Problem 30)

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{5^k} = \sum_{k=0}^{\infty}\left(\frac{-1}{5}\right)^k = 1 + \left(-\frac{1}{5}\right) + \left(-\frac{1}{5}\right)^2 + \cdots$$

It is a Geometric series with first term 1 and common ratio $-\frac{1}{5}$ and $\left|-\frac{1}{5}\right| < 1$.

$$\Rightarrow \sum_{k=0}^{\infty}\frac{(-1)^k}{5^k} = \frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{6}$$

4. (Section 11.1, Problem 31)

$$\sum_{k=0}^{\infty}\frac{1-2^k}{3^k} = \sum\frac{1}{3^k} - \sum\frac{2^k}{3^k} = \frac{3}{2} - 3 = -\frac{3}{2}$$

[Note: since both $\sum\frac{1}{3^k}$, $\sum\frac{2^k}{3^k}$ exist (ie, the sums converge) So we can separate it into two parts]

and $\displaystyle\sum_{k=0}^{\infty}\frac{1}{3^k} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$, $\displaystyle\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{1-\frac{2}{3}} = 3$

first term 1, ratio $\frac{1}{3}$; first term 1, ratio $\frac{2}{3}$

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ $\begin{cases} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$ p-series

5. (Section 11.1, Problem 33)

$$\sum_{k=0}^{\infty} \frac{2^{k+3}}{3^k} = \sum_{k=0}^{\infty} \frac{2^3 \cdot 2^k}{3^k} = 2^3 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

$$= 2^3 \cdot \frac{1}{1-\frac{2}{3}} = 8 \cdot 3 = 24$$

Geometric series
first term $\left(\frac{2}{3}\right)^0 = 1$
common ratio $\frac{2}{3}$

$\boxed{24}$

6. (Section 11.1, Problem 43)

$$\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-x)^k, \quad \text{for } |x|<1 \text{ (so is } |-x|)$$

$$= \frac{1}{1-(-x)} = \frac{1}{1+x}.$$

first term $(-x)^0 = 1$
common ratio "$-x$"

7. (Section 11.1, Problem 45)

$\frac{x}{1-x}$ for $|x|<1$ $\Rightarrow$ It is a sum of Geometric series with first term "x" and common ratio "x"

$$\Rightarrow \frac{x}{1-x} = \sum_{n=0}^{\infty} x \cdot (x)^n = \sum_{n=0}^{\infty} x^{n+1}$$

8. (Section 11.1, Problem 50)

$$\sum_{k=0}^{\infty} \frac{(-5)^k}{4^{k+1}} = \sum_{k=0}^{\infty} \frac{(-5)^k}{4 \cdot 4^k} = \sum_{k=0}^{\infty} \frac{1}{4} \cdot \left(\frac{-5}{4}\right)^k \text{ diverges.}$$

$$= \frac{1}{4} \sum_{k=0}^{\infty} \left(-\frac{5}{4}\right)^k$$

By Basic Divergence test
since $\left|\left(-\frac{5}{4}\right)^k\right| \to \infty$ as $k \to \infty$, then $\sum \left(\frac{-5}{4}\right)^k$ diverges.

9. (Section 11.2, Problem 1)

$\sum \frac{k}{k^3+1}$ $\boxed{\text{converges}}$ by Basic Comparison test

since $\sum \frac{k}{k^3} = \sum \frac{1}{k^2}$ converges (by ~~integral test~~ p-series)

and $\frac{k}{k^3+1} < \frac{k}{k^3}$, then by ______

$\boxed{k \in \mathbb{N} \Rightarrow \frac{k}{k^3+1} > 0}$

10. (Section 11.2, Problem 2)

$\sum \frac{1}{3k+2}$ $\boxed{\text{diverges}}$ by Limit Comparison test

$= a_k$, b_k

since $\sum \frac{1}{k}$ diverges (by p-series)

and $\frac{a_k}{b_k} = \frac{1}{k} \cdot \frac{3k+2}{1} \to 3 > 0$ as $k \to \infty$

Note: $|\tan^{-1} k| \le \frac{\pi}{2}$

11. (Section 11.2, Problem 4)

$\sum \frac{\ln k}{k}$ $\boxed{\text{diverges}}$ by Basic Comparison test.

Since $0 < \frac{1}{k} < \frac{\ln k}{k}$ as $k > 3$ and $\sum \frac{1}{k}$ diverges

$\boxed{D}$

12. (Section 11.2, Problem 5)

$\sum \frac{1}{\sqrt{k+1}}$ $\boxed{\text{diverges}}$ by Limit Comparison test

b_k

Since $\sum \frac{1}{\sqrt{k}}$ $= a_k$ diverges (by p-series for $\boxed{p = \frac{1}{2}} < 1$)

and $\frac{a_k}{b_k} = \frac{1}{\sqrt{k}} \cdot \frac{\sqrt{k+1}}{1} \to 1 > 0$ as $k \to \infty$ $\boxed{D}$

13. (Section 11.2, Problem 7)

$\sum \underbrace{\frac{1}{\sqrt{2k^2 - k}}}_{b_k}$ $\boxed{\text{diverges}}$ by Limit Comparison test

Since $\sum \underbrace{\frac{1}{k}}_{a_k}$ diverges and $\frac{a_k}{b_k} = \frac{1}{k} \cdot \frac{\sqrt{2k^2 - k}}{1}$

$= \sqrt{\frac{2k^2 - k}{k^2}} \to \sqrt{2} > 0$ as $k \to \infty$ $\boxed{D}$

14. (Section 11.2, Problem 9)

$\sum \frac{\tan^{-1} k}{1 + k^2}$ $\boxed{\text{Converges}}$ by Basic Comparison test

Since $\sum \frac{1}{k^2}$ converges and

$0 < \frac{\tan^{-1} k}{1 + k^2} \le \frac{\frac{\pi}{2}}{1 + k^2} < \frac{\frac{\pi}{2}}{k^2}$ ($\sum \frac{1}{k^2}$ converges implies $\frac{\pi}{2} \sum \frac{1}{k^2}$ converges)

the top is bigger

the bottom is smaller

$\boxed{C}$

15. (Section 11.2, Problem 12)

$\sum \frac{1}{k(k+1)(k+2)}$

$\boxed{\text{Converges}}$ by Basic Comparison test

Since $\sum \frac{1}{k^3}$ converges and $0 < \frac{1}{k(k+1)(k+2)} < \frac{1}{k^3}$

$\boxed{C}$

16. (Section 11.2, Problem 15)

$\sum \frac{\ln \sqrt{k}}{k} = \sum \underbrace{\frac{1}{2} \frac{\ln k}{k}}_{b_n}$ diverges by Limit Comparison test

($\sqrt{k} = k^{\frac{1}{2}}$)

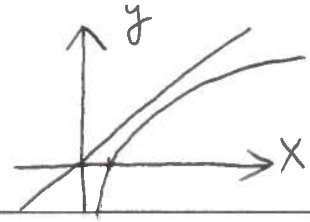
Since $\sum \underbrace{\frac{\ln k}{k}}_{a_n}$ diverges (see Q 11) and.

$\frac{a_n}{b_n} = \frac{\ln k}{k} \cdot \frac{2k}{\ln k} = 2 \to 2 > 0$ as $k \to \infty$ $\boxed{D}$

$\ln k < k$

$\Downarrow \qquad \Downarrow$

$y=\ln x \qquad y=x$



~~let $f(x) = x - \ln x$~~

or ~~$f'(x) = 1 - \frac{1}{x} > 0$~~ as $x > 0$

~~$\Rightarrow f$ is $\uparrow$~~

17. (Section 11.2, Problem 16)

$\sum \frac{2}{k(\ln k)^2}$ $\boxed{\text{converges}}$ by Integral test

Since $\int_2^\infty \frac{2\,dx}{x(\ln x)^2} = \lim_{b\to\infty} -\frac{2}{\ln x}\Big|_2^b = \frac{2}{\ln 2}$

C

18. (Section 11.2, Problem 17)

$\sum \frac{1}{2+3^{-k}}$ $\boxed{\text{diverges}}$ by Basic Divergence test

Since $\frac{1}{2+3^{-k}} \to \frac{1}{2} \neq 0$ as $k \to \infty$

D

19. (Section 11.2, Problem 20)

$\sum \frac{k^4-1}{3k^2+5}$ $\boxed{\text{diverges}}$ by Basic Divergence Test

Since $\frac{k^4-1}{3k^2+5} \to \infty \neq 0$ as $k \to \infty$

D

20. (Section 11.2, Problem 25)

$\sum \underbrace{\frac{2k+1}{\sqrt{k^4+1}}}_{b_k}$ $\boxed{\text{diverges}}$ by Limit comparison test

Since $\sum \underbrace{\frac{1}{k}}_{a_k}$ diverges and

$\frac{a_k}{b_k} = \frac{1}{k} \cdot \frac{\sqrt{k^4+1}}{2k+1} = \sqrt{\frac{k^4+1}{(2k^2+k)^2}} \to \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2} > 0$ as $k \to \infty$

D

21. (Section 11.2, Problem 27)

$\sum \underbrace{\frac{2k+1}{\sqrt{k^5+1}}}_{b_k}$ $\boxed{\text{converges}}$ by Limit comparison test

Since $\sum \underbrace{\frac{1}{k^{\frac{3}{2}}}}_{a_k}$ converges and

$\frac{a_k}{b_k} = \frac{1}{k^{\frac{3}{2}}} \cdot \frac{\sqrt{k^5+1}}{2k+1} = \sqrt{\frac{k^5+1}{(2k^{\frac{5}{2}}+k^{\frac{3}{2}})^2}} \to \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2} > 0$ as $k \to \infty$

C

22. (Section 11.2, Problem 29)

$\sum ke^{-k^2} = \sum \frac{k}{e^{k^2}}$ $\boxed{\text{converges}}$ by Integral test

Since $\int_1^\infty \frac{x}{e^{x^2}}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}\frac{1}{e^{x^2}}\right]_1^b = -0 + \frac{1}{2e} < \infty$

C

23. (Section 11.2, Problem 31)

$\sum \frac{2+\sin k}{k^2}$ $\boxed{\text{converges}}$ by Basic Comparison Test

Since $\sum \frac{3}{k^2}$ converges and

$0 < \left|\frac{2+\sin k}{k^2}\right| < \frac{3}{k^2}$

$\boxed{C}$

24. (Section 11.2, Problem 32)

$\sum \underbrace{\frac{2+\cos k}{\sqrt{k+1}}}_{b_k}$ $\boxed{\text{diverges}}$ by Limit Comparison Test

Since $\sum \underbrace{\frac{1}{\sqrt{k}}}_{a_k}$ diverges and $\frac{2+\cos k}{\sqrt{k+1}} > \underbrace{\frac{1}{\sqrt{k+1}}}_{b_k}$

$\frac{a_k}{b_k} = \frac{1}{\sqrt{k}} \cdot \frac{\sqrt{k+1}}{1} \to 1$ as $k \to \infty$

25. (Section 11.3, Problem 1)

$\sum \frac{10^k}{k!}$ Converges

Let $a_k = \frac{10^k}{k!}$. by $\boxed{\text{Ratio test}}$, we have

$\frac{a_{k+1}}{a_k} = \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k} = \frac{10}{k+1} \to 0 < 1$

as $k \to \infty$

26. (Section 11.3, Problem 2)

$\sum \frac{1}{k2^k}$ Converges

By ratio Test, let $a_k = \frac{1}{k2^k}$, then

$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)2^{k+1}} \cdot \frac{k2^k}{1} = \frac{k}{k+1} \cdot \frac{1}{2}$ $\boxed{\to \frac{1}{2} < 1}$

as $k \to \infty$

27. (Section 11.3, Problem 4)

$\sum \left(\frac{k}{2k+1}\right)^k$ Converges

By Root Test, let $a_k = \left(\frac{k}{2k+1}\right)^k$, then

$\sqrt[k]{a_k} = \frac{k}{2k+1} \to \frac{1}{2} < 1$ as $k \to \infty$

28. (Section 11.3, Problem 5)

$\sum \frac{k!}{100^k}$ Diverges.

By Ratio test, let $a_k = \frac{k!}{100^k}$, then

$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{100^{k+1}} \cdot \frac{100^k}{k!} = \cancel{k+1} \cdot \frac{k}{\cancel{k+1}} = k \to \infty > 1$

as $k \to \infty$

(*) $\lim_{k\to\infty}\sqrt[k]{k} = ?\ \lim_{k\to\infty} e^{\ln k^{\frac{1}{k}}} = e^0 = 1$

$\underbrace{\sqrt[k]{k}}_{k^{\frac{1}{k}}}$

Find $\lim_{k\to\infty}\ln k^{\frac{1}{k}} = \lim_{k\to\infty}\frac{\ln k}{k} \overset{(\frac{\infty}{\infty})}{\underset{L'}{=}} 0$

29. (Section 11.3, Problem 7)

$\sum \frac{k^2+2}{k^3+6k}$ Diverges

Since, by limit Comparison Test, let $a_k=\frac{1}{k}$

$b_k=\frac{k^2+2}{k^3+6k}$, $\frac{a_k}{b_k}=\frac{1}{k}\cdot\frac{k^3+6k}{k^2+2}\to 1 > 0$ as $k\to\infty$

and $\sum\frac{1}{k}$ diverges.

30. (Section 11.3, Problem 9)

$\sum k\left(\frac{2}{3}\right)^k$ Converges

By Root test, let $a_k=k\left(\frac{2}{3}\right)^k$, then

$\sqrt[k]{a_k}=\underbrace{\sqrt[k]{k}}_{(*)}\cdot\frac{2}{3}\to 1\cdot\frac{2}{3}=\frac{2}{3}<1$ as $k\to\infty$

31. (Section 11.3, Problem 13)

$\sum\frac{k!}{10^{4k}}$ Diverges.

By Ratio test, let $a_k=\frac{k!}{10^{4k}}$, then

$\frac{a_{k+1}}{a_k}=\frac{(k+1)!}{10^{4(k+1)}}\cdot\frac{10^{4k}}{k!}=\frac{(k+1)\cdot k!}{10^{4k}\cdot 10^4}\cdot\frac{10^{4k}}{k!}=\frac{k+1}{10^4}$

$\frac{k+1}{10^4}\to\infty>1$ as $k\to\infty$

(**) $\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k=\lim_{k\to\infty}e^{\ln\left(\frac{k}{k+1}\right)^k}=e^{-1}$

Find $\lim_{k\to\infty}\ln\left(\frac{k}{k+1}\right)^k=\lim_{k\to\infty}\frac{\ln\left(1-\frac{1}{k+1}\right)}{\frac{1}{k}}\overset{(\frac{0}{0})}{\underset{L'}{=}}\lim_{k\to\infty}\frac{\frac{k+1}{k}\cdot\left(+\frac{1}{(k+1)^2}\right)}{-\frac{1}{k^2}}=-1$

32. (Section 11.3, Problem 14)

$\sum\frac{k^2}{e^k}$ Converges.

By Ratio Test, let $a_k=\frac{k^2}{e^k}$, then

$\frac{a_{k+1}}{a_k}=\frac{(k+1)^2}{e^{k+1}}\cdot\frac{e^k}{k^2}=\frac{1}{e}\cdot\frac{k+1}{k}\to\frac{1}{e}<1$ as $k\to\infty$

Note: $e>2.71$

33. (Section 11.3, Problem 16)

$\sum\frac{2^k k!}{k^k}$ Converges

By Ratio Test, let $a_k=\frac{2^k k!}{k^k}$, then

$\frac{a_{k+1}}{a_k}=\frac{2^{k+1}(k+1)!}{(k+1)^{k+1}}\cdot\frac{k^k}{2^k\cdot k!}=\frac{2^k\cdot 2\cdot(k+1)k!}{(k+1)^k\cdot(k+1)}\cdot\frac{k^k}{2^k\cdot k!}$

$=2\left(\frac{k}{k+1}\right)^k\underset{(**)}{\to}\frac{2}{e}<1$ as $k\to\infty$

34. (Section 11.3, Problem 17)

$\sum\frac{k!}{(k+2)!}=\sum\frac{k!}{(k+2)(k+1)k!}=\sum\underbrace{\frac{1}{(k+2)(k+1)}}_{b_k}$ Converges

By Limit Comparison Test, since $\sum\underbrace{\frac{1}{k^2}}_{a_k}$ converges

and $\frac{a_k}{b_k}=\frac{1}{k^2}\cdot\frac{(k+2)(k+1)}{1}\to 1>0$

as $k\to\infty$

$k! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot (k-2)\cdot (k-1)\cdot k$

35. (Section 11.3, Problem 21)

$\sum \left(\frac{k}{k+100}\right)^k$ diverges by Basic Divergence Test.

Find $\lim_{k\to\infty} \left(\frac{k}{k+100}\right)^k = \lim_{k\to\infty} e^{\ln\left(\frac{k}{k+100}\right)^k} = e^{-100} \neq 0$

$$\boxed{\lim_{k\to\infty} \ln\left(\frac{k}{k+100}\right)^k = \lim_{k\to\infty} \frac{\ln\left(\frac{k}{k+100}\right)}{\frac{1}{k}} \left(\frac{0}{0}\right) \overset{L'}{=} \lim_{k\to\infty} \frac{\frac{k+100}{k}\cdot\frac{100}{(k+100)^2}}{-\frac{1}{k^2}} = -100}$$

38. (Section 11.3, Problem 28)

$\sum \frac{k!}{1\cdot 3 \cdots (2k-1)}$ Converges.

By Root Test, let $a_k = \frac{k!}{1\cdot 3\cdot 5 \cdots (2k-1)}$, Then

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{1\cdot 3\cdot 5\cdots (2k-1)(2k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots (2k-1)}{k!} = \boxed{\frac{k+1}{2k+1} \to \frac{1}{2} < 1 \text{ as } k\to\infty}$$

36. (Section 11.3, Problem 22)

$\sum \frac{(k!)^2}{(2k)!} \Rightarrow$ Converges

By Ratio test, let $a_k = \frac{(k!)^2}{(2k)!}$, Then

$$\frac{a_{k+1}}{a_k} = \frac{[(k+1)!]^2}{(2k+2)!} \cdot \frac{(2k)!}{(k!)^2} = \frac{(k+1)(k+1)}{(2k+2)(2k+1)} \to \boxed{\frac{1}{4} < 1 \text{ as } k\to\infty}$$

$[(k+1)!][(k+1)!]$

$(2k+2)(2k+1)\cdot(2k)!$

39. (Section 11.3, Problem 29)

$\sum \frac{2\cdot 4\cdots 2k}{(2k)!} = \sum \frac{1}{1\cdot 3\cdot 5\cdots (2k-1)}$ Converges

By Ratio Test, let $a_k = \frac{1}{1\cdot 3\cdot 5\cdots (2k-1)}$

Then $\frac{a_{k+1}}{a_k} = \frac{1}{1\cdot 3\cdot 5\cdots (2k-1)\cdot(2k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots (2k-1)}{1} = \boxed{\frac{1}{2k+1} \to 0 < 1 \text{ as } k\to\infty}$

37. (Section 11.3, Problem 26)

$\sum \frac{k!}{k^k}$ Converges.

By Ratio Test, let $a_k = \frac{k!}{k^k}$, Then

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} = (k+1)\cdot\frac{1}{k+1}\cdot \boxed{\left(\frac{k}{k+1}\right)^k = \left(\frac{k}{k+1}\right)^k} \; (**)$$

$(k+1)^k\cdot(k+1)$

$\downarrow$

$\frac{1}{e} < 1$ as $k\to\infty$

40. (Section 11.3, Problem 37)

$2^0 \quad 2^1 \quad 2^2 \quad 2^3$

$$\frac{1}{2} + \frac{2}{3^2} + \frac{4}{4^3} + \frac{8}{5^4} + \cdots = \sum_{k=0}^{\infty} \frac{2^k}{(k+2)^{k+1}}$$

2^1 $k=0 \quad k=1 \quad k=2 \quad k=3 \quad \cdots$

$$= \sum_{k=0}^{\infty} \frac{2^k}{(k+2)^k} \cdot \frac{1}{k+2} = \sum_{k=0}^{\infty} \left(\frac{2}{k+2}\right)^k \cdot \frac{1}{k+1} \quad \text{converges}$$

By root test, let $a_k = \left(\frac{2}{k+1}\right)^k \cdot \frac{1}{k+1}$

$\sqrt[k]{a_k} = \frac{2}{k+1}\cdot\frac{1}{\sqrt[k]{k+1}} \overset{(*)}{\to} 0\cdot 1 = 0 < 1$ as $k\to\infty$.